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RESEARCH ARTICLE

SOME POLYNOMIAL INEQUALITIES IN THE COMPLEX DOMAIN WITH PRESCRIBED ZEROS

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Abstract

In this paper certain polynomial inequalities with restricted zeros are given, which generalize and refine some well-known polynomial inequalities due to Ankeny and Rivlin, Dewan, Singh and Mir, Mir, Wani and Hussain and others.

Keywords: Inequalities, Restricted zeros, Polynomials, Maximum modulus.

1. Introduction.

Let $P(z)$ *be a polynomial of degree* n , $M(P,R) = \max_{|z|=R>0} |P(z)|$.

Then it is well known that

$$M(P,R) \le R^n\, M(P,1) \qquad for \quad R \ge 1 \qquad (1.1)$$

And

$$M(P,r) \ge r^n\, M(P,1) \qquad for \quad 0 < r \le 1 \qquad (1.2)$$

Inequality (1.1) is a simple deduction from Maximum Modulus Principle (see [1]), whereas inequality (1.2) is due to [2]. Both the above inequalities are sharp and an equality in each holds for the polynomials having all their zeros at the origin.

For polynomials not vanishing in $|z| < 1$, the above inequalities have been replaced by [3] and [4] repectivly

$$M(P,R) \le \left(\frac{R^n+1}{2}\right) M(P,1) \qquad for \quad R \ge 1 \qquad (1.3)$$

And

$$M(P,r) \ge \left(\frac{r+1}{2}\right)^n M(P,1) \qquad for \quad 0 < r \le 1 \qquad (1.4)$$

As generalization of (1.4), Govil [5] proved that

if $P(z)$ *has no zeros in* $|z| < 1$, *then for* $0 < r \le R \le 1$

$$M(P,r) \ge \left(\frac{1+r}{1+R}\right)^n M(P,R) \qquad (1.5)$$

Aziz and Dawood [6] further improved inequality (1.3) under the same hypothesis and Proved that

$$M(P,R) \le \left(\frac{R^n+1}{2}\right) M(P,1) - \left(\frac{R^n-1}{2}\right) \min_{|z|=1} |P(z)| \qquad for \quad R \ge 1 \qquad (1.6)$$

As an extension of (1.6) Dewan et al. [7] proved that if $P(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then for $R \ge 1$

$$M(P,R) \le \left(\frac{R^n+1}{2}\right) M(P,1) - \left(\frac{R^n-1}{2}\right) \min_{|z|=1} |P(z)| - \frac{2}{n+1}\left\{\frac{R^n-1}{n} - (R-1)\right\} |P'(0)|$$
$$- \left\{\frac{(R^n-1)-n(R-1)}{n(n-1)} - \frac{(R^{n-2}-1)-(n-2)(R-1)}{(n-2)(n-3)}\right\} |P''(0)|, \quad \text{provided} \quad n > 3 \qquad (1.7)$$

and

$$M(P,R) \le \left(\frac{R^n+1}{2}\right) M(P,1) - \left(\frac{R^n-1}{2}\right) \min_{|z|=1} |P(z)| - \frac{2}{n+1}\left\{\frac{R^n-1}{n} - (R-1)\right\} |P'(0)|$$
$$- \frac{(R-1)^n}{n(n-1)} |P''(0)|, \text{provided} \quad n = 3 \quad (1.8)$$

Recently, Govil *el at.* [8] generalized inequality (1.5) under the same hypothesis and Proved that

$$M(P,r) \ge \left(\frac{1+r}{1+R}\right)^n \left[M(P,R) + n \min_{|z|=1} |P(z)| \,\ln\left(\frac{1+R}{1+r}\right)\right],$$
$$for \quad 0 < r < R \le 1 \qquad (1.9)$$

In this paper we prove the following generalization of (1.3), (1.7) (1.8) and (1.9).

2. Main Results.

Theorem 2.1. If $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \le \mu < n$, *is a polynomial of degree* n *having no zeros in* $|z| < k$, $k \ge 1$, *then for* $R \ge 1$ *and* every positive integer s,

$$\{M(P,R)\}^s \le \left[\frac{(R^{ns} + k^{\mu+1})n|a_0| + \mu|a_\mu|(R^{ns}k^{\mu+1} + k^{2\mu})}{(1 + k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1} + k^{2\mu})}\right]\{M(P,1)\}^s \tag{2.1}$$

On taking $s = 1$, *the above theorem reduces to*

Corollary 2.1. *If* $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \le \mu < n$, is a polynomial of degree n *having no zero in* $|z| < k$, $k \ge 1$, *then for* $R \ge 1$

$$M(P,R) \le \left[\frac{(R^{n} + k^{\mu+1})n|a_0| + \mu|a_\mu|(R^{n}k^{\mu+1} + k^{2\mu})}{(1 + k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1} + k^{2\mu})}\right]M(P,1) \tag{2.2}$$

Taking $\mu = 1$, *in Theorem* 2.1, we get

Corollary 2.2. *If* $P(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *having no zeros in* $|z| < k$, $k \ge 1$, *then for* $R \ge 1$ *and* every positive integer s,

$$\{M(P,R)\}^s \le \left[\frac{(R^{ns} + k^2)n|a_0| + |a_1|\, k^2(R^{ns} + 1)}{(1 + k^2)n|a_0| + 2k^2|a_1|}\right]\{M(P,1)\}^s \tag{2.3}$$

Setting $k = 1$, *in Corallary* 2.2, gives

***Corollary* 2.3.** *If* $P(z)$ *is a polynomialof degree* n *having no zeros in* $|z| < 1$, *then for* $R \ge 1$ *and* every positive integer s,

$$\{M(P,R)\}^s \le \left[\frac{R^{ns} + 1}{2}\right]\{M(P,1)\}^s \tag{2.4}$$

Remark 2.1. *For* $s = 1$ in inequality (2.4) reduces to inequality (1.3).

***Example* 2.1.** *Consider the polynomial* $P(z) = 1000 + z^2 - z^3 - z^4$. *Clearly,*

here $\mu = 2$ *and* $n = 4$. *Since we found numerically for* $|z| < 5.43003$ *and*

thus we take $k = 5.4$. *If we take* $s = 2$, $R = 3$ *and* $M(P,1) = 1003$.

Using Theorem 2.1,

$$\{M(P,R)\}^s \le 45788667.52$$

Theorem 2.2. *If* $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \le \mu \le n$, is a polynomial of degree $n \ge 3$ *having no zeros in* $|z| < k$, $k \ge 1$, *then for* $R \ge 1$

$$\begin{aligned} M(P,R) \le {} & \left[\frac{R^n\left(n|a_0| + \mu|a_\mu|k^{\mu+1}\right) + \left(n|a_0|k^{\mu+1} + \mu|a_\mu|k^{2\mu}\right)}{(1 + k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1} + k^{2\mu})}\right]M(P,1) \\ & - \frac{(R^n - 1)}{k^n}\left[\frac{n|a_0|\,k^{\mu+1} + \mu|a_\mu|\,k^{2\mu}}{(1 + k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1} + k^{2\mu})}\right]\min_{|z|=k}|P(z)| \\ & - \frac{2\,|a_1|}{(n+1)}\left\{\frac{R^n - 1}{n} - (R-1)\right\} \\ & - |a_2|\left\{\frac{(R^n - 1) - n(R-1)}{n(n-1)} - \frac{(R^{n-2} - 1) - (n-2)(R-1)}{(n-2)(n-3)}\right\}, \text{provided } n > 3 \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} M(P,R) \le {} & \left[\frac{R^n\left(n|a_0| + \mu|a_\mu|k^{\mu+1}\right) + \left(n|a_0|k^{\mu+1} + \mu|a_\mu|k^{2\mu}\right)}{(1 + k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1} + k^{2\mu})}\right]M(P,1) \\ & - \frac{(R^n - 1)}{k^n}\left[\frac{n|a_0|\,k^{\mu+1} + \mu|a_\mu|\,k^{2\mu}}{(1 + k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1} + k^{2\mu})}\right]\min_{|z|=k}|P(z)| \\ & - \frac{2\,|a_1|}{(n+1)}\left\{\frac{R^n - 1}{n} - (R-1)\right\} \\ & - |a_2|\frac{(R-1)^n}{n(n-1)}, \text{provided } n = 3 \end{aligned} \tag{2.6}$$

If we take $\mu = 1$, *in Theorem* 2.2, we get

***Corallary* 2.4.** *If* $P(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* $n \ge 3$ *having no* zeros in $|z| < k$, $k \ge 1$, *then for* $R \ge 1$,

$$\begin{aligned} M(P,R) \le {} & \left[\frac{R^n\left(n|a_0| + |a_1|k^2\right) + \left(n|a_0| + |a_1|\right)k^2}{(1 + k^2)n|a_0| + 2|a_1|k^2}\right]M(P,1) - \\ & \frac{(R^n - 1)}{k^n}\left[\frac{(n|a_0| + |a_1|)\,k^2}{(1 + k^2)n|a_0| + 2\,|a_1|k^2}\right]\min_{|z|=k}|P(z)| \\ & - \frac{2\,|a_1|}{(n+1)}\left\{\frac{R^n - 1}{n} - (R-1)\right\} - \\ & |a_2|\left\{\frac{(R^n - 1) - n(R-1)}{n(n-1)} - \frac{(R^{n-2} - 1) - (n-2)(R-1)}{(n-2)(n-3)}\right\}, \text{provided } n > 3 \end{aligned} \tag{2.7}$$

and

$$M(P,R) \le \left[\frac{R^n\,(n|a_0|+|a_1|k^2)+(n|a_0|+|a_1|)k^2}{(1+k^2)n|a_0|+2|a_1|k^2}\right]M(P,1)-\frac{(R^n-1)}{k^n}\left[\frac{(n|a_0|\;+|a_1|\,)k^2}{(1+k^2)n|a_0|+2\,|a_1|k^2}\right]\min_{|z|=k}|P(z)| -\frac{2\,|a_1|}{(n+1)}\left\{\frac{R^n-1}{n}-(R-1)\right\} -|a_2|\frac{(R-1)^n}{n(n-1)}$$

, provided $n=3$ (2.8)

Remark 2.2. *For* $k=1$ in inequalities (2.7), (2.8) reduces to inequalities (1.7), (1.8) respectively due to Dewan et al [7].

Theorem 2.3. *If* $P(z)=a_0+\sum_{v=\mu}^{n} a_v z^v, 1\le\mu\le n$, *is a polynomial of degree n having all its zeros in $|z|\le k$, $k\le 1$, then for $1\le\lambda<\rho$*

$$M(P,\rho)\ \ge \left(\frac{k^n\rho^n\lambda^n(1+\rho^\mu)^{n/\mu}}{k^n\lambda^n(1+\rho^\mu)^{n/\mu}+\rho^n(k^\mu+\lambda^\mu)^{n/\mu}-\lambda^n(k^\mu+\rho^\mu)^{n/\mu}}\right)\left\{\frac{1}{\lambda^n}M(P,\lambda)+\frac{1}{k^n}m\ \ln\left(\frac{\rho^\mu(k^\mu+\lambda^\mu)}{\lambda^\mu(k^\mu+\rho^\mu)}\right)^{n/\mu}\right\},$$

where $m=\min_{|z|=k}|P(z)|$ (2.9)

Taking $\mu=1$ in above Theorem, we get

Corollary 2.5. *If* $P(z)=\sum_{v=0}^{n} a_v\, z^v$ *is a polynomial of degree n having all its zeros in $|z|\le k$, $k\le 1$, then for $1\le\lambda<\rho$*

$$M(P,\rho)\ge\left(\frac{k^n\rho^n\lambda^n(1+\rho)^n}{k^n\lambda^n(1+\rho)^n+\rho^n(k+\lambda)^n-\lambda^n(k+\rho)^n}\right)\left\{\frac{1}{\lambda^n}M(P,\lambda)+\frac{1}{k^n}m\ln\left(\frac{\rho(k+\lambda)}{\lambda(k+\rho)}\right)^n\right\},$$

where $m=\min_{|z|=k}|P(z)|$ (2.10)

We take λ=1 in inequality (2.9), we get

Corollary 2.6. *If* $P(z)=a_0+\sum_{v=\mu}^{n} a_v z^v$, $1\le\mu\le n$, *is a polynomial of degree n having all its zeros in $|z|\le k$, $k\le 1$, then for $1<\rho$*

$$M(P,\rho)\ge\left(\frac{k^n\rho^n(1+\rho^\mu)^{n/\mu}}{k^n(1+\rho^\mu)^{n/\mu}+\rho^n(k^\mu+1)^{n/\mu}-(k^\mu+\rho^\mu)^{n/\mu}}\right)\left\{M(P,1)+\frac{1}{k^n}m\ \ln\left(\frac{\rho^\mu(k^\mu+1)}{(k^\mu+\rho^\mu)}\right)^{n/\mu}\right\},$$

where $m=\min_{|z|=k}|P(z)|$ (2.11)

Taking $k=1$ in Theorem 2.3, we get

Corollary 2.7. *If* $P(z)=a_0+\sum_{v=\mu}^{n} a_v z^v$, $1\le\mu\le n$, *is a polynomial of degree n having all its zeros in $|z|\le 1$, then for $1\le\lambda<\rho$*

$$M(P,\rho)\ge\left(\left(\frac{1+\rho^\mu}{1+\lambda^\mu}\right)^{n/\mu}\right)\left\{M(P,\lambda)+\lambda^n m\ \ln\left(\frac{\rho^\mu(1+\lambda^\mu)}{\lambda^\mu(1+\rho^\mu)}\right)^{n/\mu}\right\},$$

where $m=\min_{|z|=1}|P(z)|$ (2.12)

Setting λ=1 in corollary 2.7, gives

Corollary 2.8. *If* $P(z)=a_0+\sum_{v=\mu}^{n} a_v z^v, 1\le\mu\le n$, *is a polynomial of degree n having all its zeros in $|z|\le 1$, then for $1<\rho$*

$$M(P,\rho)\ge\left(\left(\frac{1+\rho^\mu}{2}\right)^{n/\mu}\right)\left\{M(P,1)+m\ \ln\left(\frac{2\rho^\mu}{(1+\rho^\mu)}\right)^{n/\mu}\right\},$$

where $m=\min_{|z|=1}|P(z)|$ (2.13)

3. Lemmas.

We need the following Lemmas.

Lemma 3.1. [9] *If* $P(z)=a_0+\sum_{v=\mu}^{n} a_v z^v, 1\le\mu<n$, *is a polynomial of degree n having no zeros in $|z|<k$, $k\ge 1$, then*

$$\max_{|z|=1}|P'(z)|\le n\left(\frac{n|a_0|+\mu|a_\mu|k^{\mu+1}}{(1+k^{\mu+1})n|a_0|+\mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right)\max_{|z|=1}|P(z)| \quad (3.1)$$

Lemma 3.2. [10] *If* $P(z)=a_0+\sum_{v=\mu}^{n} a_v z^v, 1\le\mu<n$, *is a polynomial of degree n having no zero in $|z|<k$, $k\ge 1$, then*

$$\max_{|z|=1}|P'(z)|\le n\left\{\frac{n|a_0|+\mu|a_\mu|k^{\mu+1}}{(1+k^{\mu+1})n|a_0|+\mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right\}\max_{|z|=1}|P(z)| -\frac{1}{k^n}\left\{1-\frac{n|a_0|+\mu|a_\mu|k^{\mu+1}}{(1+k^{\mu+1})n|a_0|+\mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right\}\min_{|z|=k}|P(z)| \quad (3.2)$$

Lemma 3.3. [11] *If $P(z)$ is a polynomial of degree n, then for $R\ge 1$*

$$M(P,R)\le R^n\,M(P,1)-\frac{2(R^n-1)}{(n+2)}|P(0)|-\left[\frac{R^n-1}{n}-\frac{R^{n-2}-1}{(n-2)}\right]|P'(0)|,$$

provided $n>2$ (3.3)

and

$$M(P,R) \le R^n M(P,1) - \frac{(R-1)}{2}\left[(R+1)|P(0)| + (R-1)|P'(0)|\right],$$

provided $n = 2$ (3.4)

***Lemma* 3.4.** [12] *If* $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \le \mu < n$, *having no zeros in* $|z| < k$, $k \ge 1$, then for $0 < r < R \le 1$

$$M(P,r) \ge \frac{(1+r^\mu)^{n/\mu}}{(1+r^\mu)^{n/\mu} + (R^\mu + k^\mu)^{n/\mu} - (k^\mu + r^\mu)^{n/\mu}}\left\{M(P,R) + m \ln\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{n/\mu}\right\}, \text{where } m = \min_{|z|=k}|P(z)| \quad (3.5)$$

4. proof of Theorems.

Proof of Theorem 2.1.

Since $P(z)$ *has no zeros in* $|z| < k$, $k \ge 1$. *Using* Lemma 3.1, we have

$$\max_{|z|=1}|P'(z)| \le n\left[\frac{n|a_0| + \mu|a_\mu|k^{\mu+1}}{(1+k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right]\max_{|z|=1}|P(z)|$$

Appling inequality (1.1) to the polynomial $P'(z)$ *which is of degree* $(n-1)$, *for* $R \ge 1$ *and* $0 \le \theta < 2\pi$,

$$|P'(Re^{i\theta})| \le \max_{|z|=R}|P'(z)| \le R^{n-1}\max_{|z|=1}|P'(z)|$$

$$\le R^{n-1}\left[n\frac{n|a_0| + \mu|a_\mu|k^{\mu+1}}{(1+k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right]\max_{|z|=1}|P(z)| \quad (4.1)$$

Now, *for* $R \ge 1$ *and* $0 \le \theta < 2\pi$,

$$\left|\{P(Re^{i\theta}\}^s - \{P(e^{i\theta}\}^s\right| = \left|\int_1^R \frac{d}{dt}\{P(te^{i\theta}\}^s\right| \le s\int_1^R |P(te^{i\theta})|^{s-1}|P'(te^{i\theta})|dt$$

This implies that

$$|P(Re^{i\theta})|^s \le |P(e^{i\theta})|^s + \{M(P,1)\}^{s-1}\int_1^R s\, t^{ns-n}|P'(te^{i\theta})|dt$$

Which clearly gives

$$\{M(P,R)\}^s \le \{M(P,1)\}^s + \{M(P,1)\}^{s-1}\int_1^R s\, t^{ns-n}|P'(te^{i\theta})|dt$$

On applying (4.1) and noting that $R \ge 1$, *we obtain*

$$\{M(P,R)\}^s \le \{M(P,1)\}^s\left[1 + \frac{n|a_0| + \mu|a_\mu|k^{\mu+1}}{(1+k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\int_1^R nst^{ns-1}dt\right]$$

Which implies

$$\{M(P,R)\}^s \le \left[1 + \frac{n|a_0| + \mu|a_\mu|k^{\mu+1}}{(1+k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}(R^{ns} - 1)\right]\{M(P,1)\}^s$$

The above is clearly equivalent to

$$\{M(P,R)\}^s \le \left[\frac{(R^{ns}+k^{\mu+1})n|a_0| + \mu|a_\mu|(R^{ns}k^{\mu+1}+k^{2\mu})}{(1+k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right]\{M(P,1)\}^s$$

and this completes the proof.

Proof of Theorem 2.2. *Using inequality* (3.3) *of Lemma* 3.3, *we have*

for $R \ge 1$

$$M(P,R) \le R^n M(P,1) - \frac{2(R^n-1)}{(n+2)}|P(0)| - \left[\frac{R^n-1}{n} - \frac{R^{n-2}-1}{(n-2)}\right]|P'(0)|,$$

for $n > 2$.

For the polynomial $P'(z)$ *which is of degree* $(n-1) > 2$, *for* $t \ge 1$

$$\max_{|z|=t}|P'(z)| \le t^{n-1}\max_{|z|=1}|P'(z)| - \frac{2(t^{n-1}-1)}{(n+1)}|a_1| - \left[\frac{t^{n-1}-1}{(n-1)} - \frac{t^{n-3}-1}{(n-3)}\right]|a_2| \quad (4.2)$$

Now, using Lemma 3.2 in inequality (4.2) and Let $m = \min_{|z|=k}|P(z)|$, for $0 \le \theta < 2\pi$

and $R \ge 1$, *we get*

$$|P(Re^{i\theta}) - P(e^{i\theta})| \le \int_1^R |P'(te^{i\theta})|\, dt$$

$$\le \Bigg[\left(\frac{n|a_0| + \mu|a_\mu|k^{\mu+1}}{(1+k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right)M(P,1) - \frac{1}{K^n}\left\{1 - \frac{n|a_0| + \mu|a_\mu|k^{\mu+1}}{(1+k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right\}m\Bigg]\int_1^R n\, t^{n-1}dt$$
$$- \frac{2}{(n+1)}|a_1|\int_1^R (t^{n-1}-1)dt - |a_2|\int_1^R\left[\frac{t^{n-1}-1}{(n-1)} - \frac{t^{n-3}-1}{(n-3)}\right] \text{dt}$$

$$M(P,R) \le M(P,1) + \left\{\left[\frac{n|a_0| + \mu|a_\mu|k^{\mu+1}}{(1+k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right]M(P,1) - \frac{1}{k^n}\left[1 - \frac{n|a_0|\,k^{\mu+1} + \mu|a_\mu|\,k^{2\mu}}{(1+k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right] m\right\}(R^n - 1)$$

$$-\frac{2\,|a_1|}{(n+1)}\left\{\frac{R^n-1}{n} - (R-1)\right\}$$

$$-|a_2|\left\{\frac{(R^n-1) - n(R-1)}{n(n-1)} - \frac{(R^{n-2}-1)-(n-2)(R-1)}{(n-2)(n-3)}\right\}$$

$$M(P,R) \le \left[1 + \frac{(R^n-1)\left(n|a_0| + \mu|a_\mu|k^{\mu+1}\right)}{(1+k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right]M(P,1) - \frac{(R^n-1)}{k^n}\left[\frac{n|a_0|\,k^{\mu+1} + \mu|a_\mu|\,k^{2\mu}}{(1+k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right] m$$

$$-\frac{2\,|a_1|}{(n+1)}\left\{\frac{R^n-1}{n} - (R-1)\right\}$$

$$-|a_2|\left\{\frac{(R^n-1) - n(R-1)}{n(n-1)} - \frac{(R^{n-2}-1)-(n-2)(R-1)}{(n-2)(n-3)}\right\}$$

Which is equivalent to

$$M(P,R) \le \left[\frac{R^n\left(n|a_0| + \mu|a_\mu|k^{\mu+1}\right) + \left(n|a_0|k^{\mu+1} + \mu|a_\mu|k^{2\mu}\right)}{(1+k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right]M(P,1) -$$

$$\frac{(R^n-1)}{k^n}\left[\frac{n|a_0|\,k^{\mu+1} + \mu|a_\mu|\,k^{2\mu}}{(1+k^{\mu+1})n|a_0| + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right]\min_{|z|=k}|P(z)|$$

$$-\frac{2\,|a_1|}{(n+1)}\left\{\frac{R^n-1}{n} - (R-1)\right\} - |a_2|\left\{\frac{(R^n-1) - n(R-1)}{n(n-1)} - \frac{(R^{n-2}-1)-(n-2)(R-1)}{(n-2)(n-3)}\right\}, \text{provided } n > 3.$$

The inequality (2.6) follows on the same lines as that of inequality (2.5) but instead of using inequality (3.3) of Lemma 3.3, we use the inequality (3.4) of the same Lemma, and hence Theorem (2.2) is proved.

Proof of Theorem 2.3.

Since $P(z)$ *has all its zeros in* $|z| \le k$, $k \le 1$,

therefore, the polynomial $q(z) = z^n P(1/z)$ *also* has all its zeros in $|z| \ge \frac{1}{k}$.

On applying inequality (3.5) of Lemma 3.4 to the polynomial $q(z)$ *, we obtain*

$$\max_{|z|=r}|q(z)| \ge \left(\frac{(1+r^\mu)^{n/\mu}}{(1+r^\mu)^{n/\mu} + (R^\mu + 1/k^\mu)^{n/\mu} - (1/k^\mu + r^\mu)^{n/\mu}}\right)\left\{\max_{|z|=R}|q(z)| + \min_{|z|=1/k}|q(z)|\,\ln\left(\frac{R^\mu k^\mu + 1}{r^\mu k^{\mu+1}}\right)^{n/\mu}\right\} \quad (4.3)$$

Since

$$|q(z)| = |z^n P(1/z)|,\ \max_{|z|=r}|q(z)| = r^n \max_{|z|=\frac{1}{r}}|P(z)| \ \ and$$

$$\min_{|z|=1/k}|q(z)| = \frac{1}{k^n}\min_{|z|=k}|P(z)| \quad (4.4)$$

Using (4.4) in (4.3), we get

$$r^n \max_{|z|=1/r}|P(z)| \ge \left(\frac{k^n(r^\mu+1)^{n/\mu}}{k^n(r^\mu+1)^{n/\mu} + (R^\mu k^\mu + 1)^{n/\mu} - (r^\mu k^\mu + 1)^{n/\mu}}\right)\left\{R^n \max_{|z|=1/R}|P(z)| + \frac{1}{k^n}\min_{|z|=k}|P(z)|\,\ln\left(\frac{R^\mu k^\mu + 1}{r^\mu k^{\mu+1}}\right)^{n/\mu}\right\} \quad (4.5)$$

Replacing r by $1/\rho$ and R by $1/\lambda$, we obtain from (4.5)

$$M(P,\rho) \ge \left[\frac{k^n\rho^n\lambda^n(1+\rho^\mu)^{n/\mu}}{k^n\lambda^n(1+\rho^\mu)^{n/\mu} + \rho^n(k^\mu+\lambda^\mu)^{n/\mu} - \lambda^n(k^\mu+\rho^\mu)^{n/\mu}}\right]\left\{\frac{1}{\lambda^n}M(P,\lambda) + \frac{1}{k^n}m\,\ln\left(\frac{\rho^\mu(k^\mu+\lambda^\mu)}{\lambda^\mu(k^\mu+\rho^\mu)}\right)^{n/\mu}\right\}$$

and this completes proof of Theorem (2.3).

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مقالة بحثية

بعض المتباينات لمتعددة الحدود في النطاق المركب بأصفار محددة

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المُلخّص

في هذا البحث نقدم بعض المتباينات لمتعددة الحدود مع الأصفار المقيدة ، والتي هي تعميم و تحسين لبعض المتباينات المعروفة لمتعددات الحدود لكل من أنكيني وريفلين ،ديوان ،سينغ ومير، مير،واني وحسين وآخرين.

الكلمات المفتاحية: المتباينات، الأصفار المقيدة، متعددة الحدود، المقياس الأقصى.

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