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RESEARCH ARTICLE

HIGHER-ORDER CARTAN DERIVATIVES AND CURVATURE TENSOR DECOMPOSITION IN FINSLER SPACES: INSIGHTS INTO MATHEMATICAL AND PHYSICAL APPLICATIONS

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Abstract

This paper delves into the intricate structure of curvature tensors within the realm of Finsler geometry. By harnessing the power of higher-order Cartan derivatives, we introduce a novel decomposition scheme for curvature tensors. This innovative approach not only provides deeper insights into the geometric properties of Finsler spaces but also establishes a foundational framework for further investigations. Our findings reveal that the proposed decomposition is instrumental in unraveling the connections between curvature, torsion, and the underlying metric structure. Moreover, we demonstrate the applicability of our results to various subdomains of Finsler geometry, including Finsler information geometry and Finsler cosmology.

Keywords: Finsler space, Cartan's covariant derivative expansion, Curvature tensor, Identities, Geometric properties.

1. Introduction

Finsler geometry, as a generalization of Riemannian geometry, offers a flexible framework for modeling diverse physical phenomena characterized by anisotropic and position-dependent metrics. Central to the study of Finsler geometry are curvature tensors, which encapsulate the intrinsic curvature properties of the underlying space. While significant progress has been made in understanding curvature tensors in Riemannian geometry, their counterparts in Finsler geometry exhibit a richer and more complex structure. Traditional approaches to analyzing curvature tensors in Finsler geometry often rely on the concept of Cartan connection. However, these methods can become cumbersome when dealing with higher-order geometric quantities. In this paper, we propose a novel approach that leverages the power of higher-order Cartan derivatives to systematically decompose curvature tensors.

This decomposition not only simplifies the analysis of curvature but also reveals new connections between curvature, torsion, and the metric structure.

The study of curvature tensors and recurrent structures in Finsler geometry and relativistic space-times has attracted considerable attention due to its significance in both mathematical theory and physical applications. The foundational work by [23] laid the groundwork for modern developments in Finsler geometry, introducing essential geometric structures and curvature concepts.

Subsequent efforts have focused on the refinement and generalization of curvature tensors. Notably, [2] and [3] investigated the properties of the w-curvature tensor and its implications in relativistic contexts. [1] extended this line of inquiry by analyzing the w*-curvature tensor on relativistic space-times, offering new insights into its geometric and physical interpretations.

In the context of Finsler geometry, a substantial body of work has been contributed by Al-Qashbari and collaborators. Their investigations have spanned a wide range of topics, including M-projective curvature tensors [4], higher-order generalized recurrent structures [6], [7], and the decomposition of generalized curvature tensors [5] and [9]. These studies have made significant strides in classifying and understanding complex recurrent phenomena in Finsler manifolds through the use of Cartan and Berwald derivatives.

Further contributions have examined special curvature structures such as the conharmonic, Weyl, and R-projective tensors, particularly in relation to their behavior under various covariant and Lie derivatives [10], [11], [12], [13], [14], [15], [17] and [18]. These efforts have helped elucidate the deeper algebraic and differential properties of curvature tensors in generalized fifth recurrent Finsler spaces (GBK-5RFn), as well as their role in broader geometric frameworks.

The work of [20] and [21], has further contextualized the behavior of higher-order recurrent Finsler spaces, emphasizing the role of Berwald-type structures and systematizing various special cases of interest. Similarly, the investigations by [22] into generalized H-recurrent spaces have provided a comparative foundation for the study of trirecurrent and BK-recurrent geometries, as also explored in later works [16], and [19].

Collectively, these studies demonstrate a growing research interest in extending classical geometric concepts through higher-order and generalized structures in Finsler spaces. The present work builds upon this foundation by further analyzing [insert specific topic of your paper here, e.g., concircular motions using Cartan's fourth curvature tensor in the Berwald sense], thereby contributing new insights into the ongoing development of Finsler geometry and its recurrent frameworks.

Two vectors y_i and y^i meet the following conditions

a) $y_i = g_{ij}\, y^j$,

b) $y_i\, y^i = F^2$,

c) $\delta^k_j y^j = y^k$,

d) $g_{ir}\, \delta^i_j = g_{rj}$,

e) $g^{jk}\delta^i_k = g^{ji}$,

f) $\dot{\partial}_j\, y^j = 1$ and

g) $\dot{\partial}_k\, y_j = g_{jk}$. (1.1)

The quantities g_{ij} and g^{ij} are covariant constant with respect to h-covariant derivative by

a) $g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 , & if \quad i = k , \\ 0 , & if \quad i \neq k . \end{cases}$

b) $g^{jk}{}_{|h} = 0$ and

c) $g_{ij|h} = 0$. (1.2)

Tensor C_{ijk} is known as (h)hv-torsion tensor defined by

$$C_{ijk} = \tfrac{1}{2}\dot{\partial}_i\, g_{jk} = \tfrac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k\, F^2 , \tag{1.3}$$

The (v)hv-torsion tensor C^h_{ik} and tensor C_{ijk} are given by

a) $C^i_{jk}\, y^j = C^i_{jk} y^k = 0$,

b) $C_{ijk}\, y^i = C_{ijk}\, y^j = C_{ijk}\, y^k = 0$.

c) $g^{jk} C_{ijk} = C_i$ and

d) $g^{jk} C_{ijh} = C^k_{ih}$. (1.4)

The vector y^i and metric function F are vanished identically for Cartan's covariant derivative.

a) $F_{|h} = 0$ and

b) $y^i{}_{|h} = 0$. (1.5)

Cartan [7] deduced the covariant derivatives of an arbitrary vector field X^i with respect to x^k which given by

$$X^i\big|_k = \dot{\partial}_k X^i + X^r\, C^{\,i}_{rk} . \tag{1.6}$$

and

$$X^i_{|k} = \partial_k\, X^i - \left(\dot{\partial}_r\, X^i\right) G^{\,r}_{k} + X^r \Gamma^{*\,i}_{rk} , \tag{1.7}$$

where the function $\Gamma^{*\,i}_{rk}$ is defined by

$$\Gamma^{*i}_{rk} = \Gamma^{\,i}_{rk} - C^{\,i}_{mr}\, \Gamma^{\,m}_{sk}\, y^s .$$

The functions Γ^{*i}_{rk} and G^r_k are connected by

$$G^{\,r}_{k} = \Gamma^{*r}_{sk}\, y^s \text{ , where } \partial_j = \frac{\partial}{\partial x^j} ,$$

$$\dot{\partial}_j = \frac{\partial}{\partial y^j} , \quad G^i_j = \dot{\partial}_j G^i .$$

In the context of Finsler geometry, equations (1.6) and (1.7) define two distinct forms of covariant differentiation: the v-covariant differentiation, also referred to as Cartan's first kind covariant derivative, and the h-covariant differentiation, known as Cartan's second kind covariant derivative. Accordingly, the notations $X^i|_k$ and $X^i_{|k}$ represent the v-covariant and h-covariant derivatives of a vector field X^i, respectively.

In this study, we present a rigorous mathematical exposition of key tensorial structures, namely the generalized curvature tensor W^i_{jkh} , the torsion tensor W^i_{jk} , and the deviation tensor W^i_j. These tensors play a fundamental role in differential geometry and theoretical physics, especially in the characterization of the curvature and torsional behavior of differentiable manifolds.

Precise definitions of these tensors are given as follows:

$$W^i_{jkh} = H^i_{jkh} + \frac{2\,\delta^i_j}{(n+1)} H_{[hk]} + \frac{2\,y^i}{(n+1)} \dot{\partial}_j H_{[kh]} + \frac{\delta^i_k}{(n^2-1)}\left(n\, H_{jh} + H_{hj} + y^r \dot{\partial}_j H_{hr}\right) - \frac{\delta^i_h}{(n^2-1)}\left(n\, H_{jk} + H_{kj} + y^r \dot{\partial}_j H_{kr}\right) , \tag{1.8}$$

$$W^i_{jk} = H^i_{jk} + \frac{y^i}{(n+1)} H_{[jk]} + 2\left\{\frac{\delta^i_{[j}}{(n^2-1)}\left(n\, H_{k]} - y^r H_{k]\,r}\right)\right\} \text{ and} \tag{1.9}$$

$W_j^i = H_j^i - H\,\delta_j^i - \frac{1}{(n+1)}\left(\dot\partial_r H_j^r - \dot\partial_j H\right) y^i$,

respectively. (1.10)

These tensors satisfy the following intrinsic identities:

a) $W_{jkh}^i\, y^j = W_{kh}^i$,

b) $W_{kh}^i\, y^k = W_h^i$,

c) $W_{jki}^i = W_{jk}$,

d) $g_{ir}\, W_{jkh}^i = W_{rjkh}$,

e) $W_{jkh}^i = -\, W_{jhk}^i$ and

f) $W_{jkh}^i + W_{khj}^i + W_{hjk}^i = 0$. (1.11)

Additionally, under specific conditions, the deviation tensor W_j^i satisfies:

a) $W_k^i\, y^k = 0$,

b) $W_i^i = 0$,

c) $g_{ir}\, W_j^i = W_{rj}$,

d) $g^{jk} W_{jk} = W$ and

e) $W_{jk}\, y^k = 0$. (1.12)

Notably, the tensor $\mathrm{W}_{\mathrm{jkh}}^{\mathrm{i}}$ is skew-symmetric with respect to the indices k and h. The discussion is further extended to Cartan's third curvature tensor R_{jkh}^i , the Ricci tensor R_{jk} , the curvature vector H_k, and the scalar curvature H . These entities are central in the geometric interpretation of manifold curvature. Through their analytical interrelations and algebraic properties, this work contributes to a deeper understanding of geometric structures in differential geometry and their implications in physical theories.

a) $R_{jkh}^i = \Gamma_{hjk}^{*i} + \left(\Gamma_{ljk}^{*i}\right) G_h^l + C_{jm}^i\left(G_{kh}^m - G_{kl}^m\, G_h^l \right) + \Gamma_{mk}^{*i}\, \Gamma_{jh}^{*m} - k/h$,

b) $R_{jkh}^i\, y^j = H_{kh}^i$,

c) $R_{jk}\, y^j = H_k$,

d) $R_{jk}\, y^k = R_j$,

e) $R_i^i = R$,

f) $g_{ir}\, R_{jkh}^i = R_{rjkh}$,

g) $R_{jkh}^i = -\, R_{jhk}^i$,

h) h) $g^{jk} R_{jkh}^i = R_h^i$,

i) $R_{jki}^i = R_{jk}$,

j) $H_i\, y^i = H_i^i = (n-1)\, H$,

k) $H_{kh}^i\, y^k = H_h^i$ and

l) $H_{ki}^i = H_k$. (1.13)

Cartan's covariant derivative, when applied to the tensors R_{jk} and δ_h^k , yields the following relations:

a) $R_{jk|m} = \lambda_m R_{jk}$ and

b) $\delta_{h|m}^k = 0$. (1.14)

Additionally, the covariant derivative with respect to $|m$ satisfies:

a) $(\delta_h^k\, R_{ij})_{|m} = \lambda_m \delta_h^k\, R_{ij}$,

b) $(g_{ij}\, R_h^k)_{|m} = \lambda_m\, g_{ij}\, R_h^k$,

c) $\left(R\delta_k^h g_{ij}\right)_{|m} = \lambda_m\, R\, \delta_k^h g_{ij}$ and

d) $\left(RR_{ij}\right)_{|m} = \lambda_m RR_{ij}$. (1.15)

Moreover, Cartan's covariant derivatives of the tensor fields T_{ijk}^h , T_{ij}^h and T_i^h with respect to x^m are given by:

a) $T_{jkh|m}^i = \lambda_m T_{jkh}^i$,

b) $T_{jk|m}^i = \lambda_m T_{jk}^i$ and

c) $T_{j|m}^i = \lambda_m T_j^i$. (1.16)

The structure of the present work is organized as follows. After the introductory Section 1, which includes foundational concepts and notations, Section 2 explores the algebraic and geometric relations between the Weyl projective curvature tensor and several other curvature tensors. Section 3 is devoted to the expansion of Cartan's covariant derivative applied to general curvature tensors. Finally, Section 4 investigates a set of tensorial identities derived in Section 2 using the expansions developed earlier.

2. Preliminaries

In Finsler geometry, a central theme is the complex interplay between various types of curvature tensors. These interrelations are frequently captured through concise and elegant mathematical identities. This paper emphasizes the study of the structural connections between the Weyl projective curvature tensor and other fundamental curvature tensors, aiming to shed light on their underlying geometric significance.

2.1. The Riemanniian Curvature Tensor R_{jkh}^i

The Riemann curvature tensor is a central construct in differential geometry, serving as a measure of the intrinsic curvature of a Riemannian manifold. It quantifies how the manifold's geometry deviates locally from that of flat Euclidean space, making it a vital invariant in understanding geometric structures. More precisely, the Riemann tensor captures the non-commutativity of second-order covariant derivatives and thus encapsulates the failure of local flatness in curved spaces.

A Riemannian manifold is said to be flat if and only if its Riemann curvature tensor vanishes identically, implying that the manifold is locally isometric to Euclidean space. Importantly, this tensor is not limited to Riemannian geometry alone; it extends naturally to pseudo-Riemannian manifolds and more generally to manifolds endowed with an affine connection.

In the broader scope of mathematical physics, the Riemann curvature tensor plays a foundational role in the general theory of relativity, where it describes the curvature of spacetime resulting from mass-energy distributions. Its components are directly involved in the formulation of Einstein's field equations.

Closely associated with the Riemann curvature tensor is the Weyl projective curvature tensor, which generalizes the notion of projective and conharmonic curvature tensors.

Definition 2.1. Weyl projective curvature tensor in terms of Riemannian curvature tensor R^i_{jkh} is defined as [12] and [21].

$$W^i_{jkh} = R^i_{jkh} + \frac{1}{(n-1)}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) . \tag{2.1}$$

In (V_4, F), we have

$$R^i_{jkh} = W^i_{jkh} - \frac{1}{3}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) . \tag{2.2}$$

2.2. Projective Curvature Tensor $\overline{W}^i_{jkh}$

The $\overline{W}$-projective curvature tensor is a geometric object of significant interest in differential geometry. It has been investigated in various geometric contexts, including Riemannian geometry, Kähler geometry, and cosmology, due to its ability to reveal subtle geometric and physical properties.

Furthermore, the concept of an M-projective curvature tensor was introduced by Pokhariyal and Mishra (1970), providing a more generalized framework for curvature analysis in Riemannian and related geometries. The definition and properties of this tensor have been explored extensively and will be discussed in the following sections of this work.

$$\overline{W}(X,Y,Z,T) = \bar{R}(X,Y,Z,T) - \frac{1}{2(n-1)}[S(Y,Z)g(X,T) - S(X,Z)g(Y,T) + g(Y,Z)S(X,T) - g(X,T)S(Y,Z)] . \tag{2.3}$$

Where: $\overline{W}(X,Y,Z,T) = g(W(X,Y)Z,T)$ and $\bar{R}(X,Y,Z,T) = g(R(X,Y)Z,T)$. (2.4)

R is the Riemann curvature tensor, S is the Ricci tensor, g is the metric tensor, n is the dimension of the manifold.

The $\overline{W}$-projective curvature tensor has a number of interesting properties. For example, it is invariant under conformal transformations. This means that it is the same for two metrics that are conformally equivalent. The $\overline{W}$-projective curvature tensor also vanishes if and only if the manifold is Ricci-flat.

The $\overline{W}$-projective curvature tensor has been used to study a variety of geometric problems. For example, it has been used to classify Riemannian manifolds, to study the geometry of Kähler manifolds, and to develop new models of gravity.

The local coordinates expression of equation (2.3) as follows

$$\overline{W}_{ljkh} = R_{ljkh} - \frac{1}{2(n-1)}\left[R_{jk} g_{lh} - R_{lk} g_{jh} + g_{jk} R_{lh} - g_{lk} R_{jh}\right] . \tag{2.5}$$

Assuming $n = 4$ and using (2.2) in equation (2.5) and contracting with g^{li}, the M-projective curvature tensor is given by

$$\overline{W}^i_{jkh} = W^i_{jkh} - \frac{1}{6}\left(\delta^i_h R_{jk} + \delta^i_k R_{jh} - g_{jk} R^i_h - g_{jh} R^i_k\right) . \tag{2.6}$$

2.3. Conformal Curvature Tensor C^i_{jkh}

The conformal curvature tensor, also known as the Weyl conformal curvature tensor, is a geometric object introduced in differential geometry. It is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. Like the Riemann curvature tensor, the Weyl tensor expresses the tidal force that a body feels when moving along a geodesic. The Weyl tensor differs from the Riemann curvature tensor in that it does not convey information on how the volume of the body changes, but rather only how the shape of the body is distorted by the tidal force.

Definition 2.2. The Conformal curvature tensor C^h_{ijk} expressed as follows

$$C^i_{jkh} = R^i_{jkh} - \frac{1}{2}\left(\delta^i_k R_{jh} - \delta^i_h R_{jk} + R^i_k g_{jh} - R^i_h g_{jk}\right) - \frac{1}{6} R\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) . \tag{2.7a}$$

Using (2.2) in equation (2.7a), we get

$$C^i_{jkh} = W^i_{jkh} - \frac{5}{6}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) - \frac{1}{6} R\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{2}\left(\delta^i_h R_{jk} - R^i_k g_{jh}\right) . \tag{2.7b}$$

2.4. Conharmonic Curvature Tensor L^i_{jkh}

The conharmonic curvature tensor is a geometric object introduced in differential geometry. It generalizes the projective curvature tensor and the conformal curvature tensor. It has been studied in a variety of contexts, including Riemannian geometry, Kähler geometry, and cosmology.

Definition 2.3. For V_4 the Conharmonic curvature tensor L^i_{jkh} defined as [10] and [18]

$$L^i_{jkh} = R^i_{jkh} - \frac{1}{2}\left(g_{jk}R^i_h + \delta^i_h R_{jk} - \delta^i_k R_{jh} - g_{jh}R^i_k\right) . \quad (2.8a)$$

Using (2.2) in equation (2.8a), we get

$$L^i_{jkh} = W^i_{jkh} + \frac{1}{6}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) - \frac{1}{2}\left(\delta^i_h R_{jk} - R^i_k g_{jh}\right) . \quad (2.8b)$$

2.5. Concircular Curvature Tensor M^i_{jkh}

The concircular curvature tensor is a geometric object introduced in differential geometry. It is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. It is closely related to the conformal curvature tensor (also known as the Weyl curvature tensor) and the projective curvature tensor. The concircular curvature tensor vanishes if and only if the manifold is concircularly flat.

Definition 2.4. The Concircular curvature tensor M_{hijk}, for V_4 is defined as [3]

$$M^i_{jkh} = R^i_{jkh} - \frac{1}{12}R\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right) . \quad (2.9)$$

Using (2.2) in equation (2.9), we get

$$M^i_{jkh} = W^i_{jkh} - \frac{1}{12}R\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right) - \frac{1}{6}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right). \quad (2.10)$$

2.6. P_1-Curvature Tensor

The P_1-curvature tensor is a geometric object introduced in differential geometry. It is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. It is closely related to the Ricci curvature tensor and the scalar curvature. The P_1-curvature tensor vanishes if and only if the manifold is Ricci-flat and has constant scalar curvature. The tensor $P_1(X,Y,Z,T)$ has been defined as

$$P_1(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{2(n-1)}[g(Y,Z)Ric(X,T) - g(Y,T)Ric(X,Z) - g(X,Z)Ric(Y,T) + g(X,T)Ric(Y,Z) . \quad (2.11)$$

We consider the P_1-curvature tensor in the index notation as [8]

$$P_{1_{ljkh}} = R_{ljkh} + \frac{1}{2(n-1)}\left[g_{jk}R_{lh} - g_{jh}R_{lk} - g_{lk}R_{jh} + g_{lh}R_{jk}\right] . \quad (2.12)$$

This can be written as

$$P^i_{1\,jkh} = R^i_{jkh} + \frac{1}{2(n-1)}\left[g_{jk}R^i_h - g_{jh}R^i_k - \delta^i_k R_{jh} + \delta^i_h R_{jk}\right] . \quad (2.13)$$

In (V_4, F), and using (2.2) in equation (2.13), we get

$$P^i_{1\,jkh} = W^i_{jkh} + \frac{1}{6}\left[\delta^i_h R_{jk} - g_{jh}R^i_k\right] - \frac{1}{3}\left[\delta^i_k R_{jh} - g_{jk}R^i_h\right] . \quad (2.14)$$

3. Expansion Curvatures Tensors in Finsler Space

The expansion curvature tensor W^i_{jkh} is a geometric object introduced in Finsler geometry.

It is a measure of the curvature of a Finsler manifold, which is a generalization of a Riemannian manifold. The expansion curvature tensor is closely related to the Weyl projective curvature tensor and the Cartan's curvature tensor. It vanishes if and only if the Finsler manifold is flat. we introduced the generalized by Cartan's covariant derivative for any tensor W^h_{ijk} was given by

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) . \quad (3.1)$$

We can write (3.1) by the follows form

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \gamma_m\left[W^i_h(0) - W^i_k(0)\right] .$$

From (1.4b) the above equation can be written as

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \gamma_m\left[W^i_h C_{ijk} y^i - W^i_k C_{ijh} y^i\right] . \quad (3.2)$$

Using (1.3) in (3.2), we get

$$W^i_{(jkh|m)} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}\gamma_m\left[W^i_h \dot\partial_k \dot\partial_i \dot\partial_j F^2 y^i - W^i_k \dot\partial_h \dot\partial_j \dot\partial_i F^2 y^i\right]. \quad (3.3)$$

Applying (1.1f) on (3.3), we get

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}\gamma_m\left[W^i_h \dot\partial_k \dot\partial_j F^2 - W^i_k \dot\partial_h \dot\partial_j F^2\right] .$$

From (1.1b) the above equation can be written as

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}\gamma_m\left[W^i_h \dot\partial_k \dot\partial_j y^j y_j - W^i_k \dot\partial_h \dot\partial_j y^j y_j\right] . \quad (3.4)$$

Applying (1.1f) again on (3.4), we get

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}\gamma_m\left[W^i_h \dot\partial_k y_j - W^i_k \dot\partial_h y_j\right] .$$

From (1.1g), we have

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$$

$$+\frac{1}{4}\gamma_m\left[W_h^i g_{jk} - W_k^i g_{jh}\right].$$

From the previous steps, we can conclude the following theorem

Theorem 3.1. The expansion of (1.16) is given by (3.5).

The dimensionality of many curvatures tensors operators will be extended in accordance with theorem 3.1.

4. Investigate the Expansion by Identities

Mathematical identities are equations that are always true, regardless of the values of the variables involved. They can be used to simplify expressions, solve equations, and prove theorems. We investigated the expansion of Cartan's covariant derivative for any curvature tensor that was given in last equation in section 3, i.e.

$$W_{jkh|m}^i = \lambda_m W_{jkh}^i + \mu_m\left(\delta_h^i g_{jk} - \delta_k^i g_{jh}\right)$$
$$+\frac{1}{4}\gamma_m\left[W_h^i g_{jk} - W_k^i g_{jh}\right]. \quad (4.1)$$

We suppose that (4.1) holds to investigate the following identities

4-1. Using Cartan's covariant derivative, we derive the following expression for equation (2.2)

$$R_{jkh|m}^i = W_{jkh|m}^i - \frac{1}{3}\gamma_m\left(\delta_k^i R_{jh} - R_h^i g_{jk}\right)_{|m}. \quad (4.2)$$

From (1.15a), (1.15b), (4.1) and (4.2), we get

$$R_{jkh|m}^i = \lambda_m W_{jkh}^i + \mu_m\left(\delta_h^i g_{jk} - \delta_k^i g_{jh}\right)$$
$$+\frac{1}{4}\gamma_m\left[W_h^i g_{jk} - W_k^i g_{jh}\right] - \frac{1}{3}\lambda_m\left(\delta_k^i R_{jh} - R_h^i g_{jk}\right).$$

This gives

$$R_{jkh|m}^i = \lambda_m\left[W_{jkh}^i - \frac{1}{3}\left(\delta_k^i R_{jh} - R_h^i g_{jk}\right)\right]$$
$$+\mu_m\left(\delta_h^i g_{jk} - \delta_k^i g_{jh}\right) + \frac{1}{4}\gamma_m\left[W_h^i g_{jk} - W_k^i g_{jh}\right]. \quad (4.3)$$

By using (2.2) in (4.3), we have

$$R_{jkh|m}^i = \lambda_m R_{jkh}^i + \mu_m\left(\delta_h^i g_{jk} - \delta_k^i g_{jh}\right)$$
$$+\frac{1}{4}\gamma_m\left[W_h^i g_{jk} - W_k^i g_{jh}\right]. \quad (4.4)$$

From the previous steps, we can conclude the following theorem

Theorem 4.1: The expansion derivative for Cartan of Riemannian curvature tensor R_{jkh}^i (2.2) is satisfies the equation (4.4).

Transvecting condition to a higher dimensional space (4.4) by y^j, using the conditions (1.5b), (1.13b) and (1.1a), we get

$$H_{kh|m}^i = \lambda_m H_{kh}^i + \mu_m\left(\delta_h^i y_k - \delta_k^i y_h\right)$$
$$+\frac{1}{4}\gamma_m\left[W_h^i y_k - W_k^i y_h\right]. \quad (4.5)$$

Again, transvecting condition to a higher dimensional space (4.5) by y^k, using the conditions (1.5b), (1.13j), (1.12a), (1,1b) and (1.1c), we get

$$H_{h|m}^i = \lambda_m H_h^i + \mu_m\left(\delta_h^i F^2 - y^i y_h\right) + \frac{1}{4}\gamma_m W_h^i F^2. \quad (4.6)$$

Therefore, the proof of theorem is completed, we can say

Theorem 4.2. In covariant derivative for Cartan's of first order for torsion tensor H_{kh}^i and deviation tensor H_h^i are given by (4.5) and (4.6).

4-2. Using Cartan's covariant derivative, we derive the following expression for equation (2.6)

$$\bar{W}_{jkh|m}^i = W_{jkh|m}^i - \frac{1}{6}\left(\delta_h^i R_{jk} + \delta_k^i R_{jh} - g_{jk} R_h^i - g_{jh} R_k^i\right)_{|m}. \quad (4.7)$$

From (1.15a), (1.15b), (4.1) and (4.7), we get

$$\bar{W}_{jkh|m}^i = \lambda_m W_{jkh}^i + \mu_m\left(\delta_h^i g_{jk} - \delta_k^i g_{jh}\right)$$
$$+\frac{1}{4}\gamma_m\left[W_h^i g_{jk} - W_k^i g_{jh}\right]$$
$$-\frac{1}{6}\lambda_m\left(\delta_h^i R_{jk} + \delta_k^i R_{jh} - g_{jk} R_h^i - g_{jh} R_k^i\right).$$

This can be written as

$$\bar{W}_{jkh|m}^i = \lambda_m\left[W_{jkh}^i - \frac{1}{6}\left(\delta_h^i R_{jk} + \delta_k^i R_{jh} - g_{jk} R_h^i - g_{jh} R_k^i\right)\right] + \mu_m\left(\delta_h^i g_{jk} - \delta_k^i g_{jh}\right)$$
$$+\frac{1}{4}\gamma_m\left[W_h^i g_{jk} - W_k^i g_{jh}\right]. \quad (4.8)$$

From (2.6) and (4.8), we have

$$\bar{W}_{jkh|m}^i = \lambda_m \bar{W}_{jkh}^i + \mu_m\left(\delta_h^i g_{jk} - \delta_k^i g_{jh}\right)$$
$$+\frac{1}{4}\gamma_m\left[W_h^i g_{jk} - W_k^i g_{jh}\right]. \quad (4.9)$$

So, the proof of theorem is completed, we can say

Theorem 4.3. The expansion derivative for Cartan's of projective curvature tensor $\bar{W}_{jkh}^i$ (2.6) is satisfies the equation (4.9).

4-3. Using Cartan's covariant derivative, we derive the following expression for equation (2.7b)

$$C_{jkh|m}^i = W_{jkh|m}^i - \frac{5}{6}\left(\delta_k^i R_{jh} - R_h^i g_{jk}\right)_{|m}$$
$$-\frac{1}{6}\left(R\left(\delta_h^i g_{jk} - \delta_k^i g_{jh}\right)\right)_{|m}$$
$$+\frac{1}{2}\left(\delta_h^i R_{jk} - R_k^i g_{jh}\right)_{|m}. \quad (4.10)$$

From (1.15a), (1.15b), (1.15d), (4.1) and (4.10), we get

$$C_{jkh|m}^i = \lambda_m W_{jkh}^i + \mu_m\left(\delta_h^i g_{jk} - \delta_k^i g_{jh}\right)$$
$$+\frac{1}{4}\gamma_m\left[W_h^i g_{jk} - W_k^i g_{jh}\right] + \frac{1}{2}\lambda_m\left(\delta_h^i R_{jk} - R_k^i g_{jh}\right)$$
$$-\frac{5}{6}\lambda_m\left(\delta_k^i R_{jh} - R_h^i g_{jk}\right) - \frac{1}{6}\lambda_m R\left(\delta_h^i g_{jk} - \delta_k^i g_{jh}\right).$$

Or, we can write as

$$C^i_{jkh|m} = \lambda_m \Big(W^i_{jkh} - \tfrac{5}{6}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) - \tfrac{1}{6} R\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{2}\left(\delta^i_h R_{jk} - R^i_k g_{jh}\right)\Big)$$

$$+ \mu_m\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right) + \tfrac{1}{4}\gamma_m\left[W^i_h g_{jk} - W^i_k g_{jh}\right]. \quad (4.11)$$

By using (2.7b) in (4.11), we have

$$C^i_{jkh|m} = \lambda_m C^i_{jkh} + \mu_m\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right) + \tfrac{1}{4}\gamma_m\left[W^i_h g_{jk} - W^i_k g_{jh}\right]. \quad (4.12)$$

In conclusion the proof of theorem is completed, we can determine

Theorem 4.4. The expansion derivative for Cartan's of Conformal curvature tensor C^h_{ijk} (2.7b) is satisfies the equation (4.12).

4-4. Using Cartan's covariant derivative, we derive the following expression for equation (2.8b)

$$L^i_{jkh|m} = W^i_{jkh|m} + \tfrac{1}{6}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right)_{|m} - \tfrac{1}{2}\left(\delta^i_h R_{jk} - R^i_k g_{jh}\right)_{|m}. \quad (4.13)$$

From (1.15a), (1.15b), (4.1) and (4.13), we get

$$L^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{4}\gamma_m\left[W^i_h g_{jk} - W^i_k g_{jh}\right] + \tfrac{1}{6}\lambda_m\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) - \tfrac{1}{2}\lambda_m\left(\delta^i_h R_{jk} - R^i_k g_{jh}\right).$$

Or can be written as

$$L^i_{jkh|m} = \lambda_m\Big[W^i_{jkh} + \tfrac{1}{6}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right) - \tfrac{1}{2}\left(\delta^i_h R_{jk} - R^i_k g_{jh}\right)\Big] + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{4}\gamma_m\left[W^i_h g_{jk} - W^i_k g_{jh}\right]. \quad (4.14)$$

From (2.8b) and (4.14), we get

$$L^i_{jkh|m} = \lambda_m L^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{4}\gamma_m\left[W^i_h g_{jk} - W^i_k g_{jh}\right]. \quad (4.15)$$

Thus, the proof of theorem is completed, we get

Theorem 4.5. The expansion derivative for Cartan of Conharmonic curvature tensor L^i_{jkh} (2.8b) is satisfies the equation (4.15).

4-5. Using Cartan's covariant derivative, we derive the following expression for equation (2.10)

$$M^i_{jkh|m} = W^i_{jkh|m} - \frac{1}{12}\left(R\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right)\right)_{|m} - \tfrac{1}{6}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right)_{|m}. \quad (4.16)$$

From (1.15a), (1.15b), (1.15d), (4.1) and (4.16), we get

$$M^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{4}\gamma_m\left[W^i_h g_{jk} - W^i_k g_{jh}\right] - \tfrac{1}{12}\lambda_m R\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right) - \tfrac{1}{6}\lambda_m\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right).$$

Or can be written as

$$M^i_{jkh|m} = \lambda_m\Big[W^i_{jkh} - \tfrac{1}{12}R\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right) - \tfrac{1}{6}\left(\delta^i_k R_{jh} - R^i_h g_{jk}\right)\Big] + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{4}\gamma_m\left[W^i_h g_{jk} - W^i_k g_{jh}\right]. \quad (4.17)$$

From (2.10) and (4.17), we have

$$M^i_{jkh|m} = \lambda_m M^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{4}\gamma_m\left[W^i_h g_{jk} - W^i_k g_{jh}\right]. \quad (4.18)$$

In conclusion the proof of theorem is completed, we can determine

Theorem 5.6. The expansion derivative for Cartan of Concircular curvature tensor M^i_{jkh} (2.10) is satisfies the equation (4.18).

4-6. Using Cartan's covariant derivative, we derive the following expression for equation (2.14)

$$P^i_{1\,jkh|m} = W^i_{jkh|m} + \tfrac{1}{6}\left(\delta^i_h R_{jk} - g_{jh} R^i_k\right)_{|m} - \tfrac{1}{3}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m}. \quad (4.19)$$

From (1.15a), (1.15b), (4.1) and (4.19), we get

$$P^i_{1\,jkh|m} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{4}\gamma_m\left[W^i_h g_{jk} - W^i_k g_{jh}\right] + \tfrac{1}{6}\lambda_m\left[\delta^i_h R_{jk} - g_{jh} R^i_k\right] - \tfrac{1}{3}\lambda_m\left[\delta^i_k R_{jh} - g_{jk} R^i_h\right]$$

Or can be written as

$$P^i_{1\,jkh|m} = \lambda_m\Big[W^i_{jkh} + \tfrac{1}{6}\left[\delta^i_h R_{jk} - g_{jh} R^i_k\right] - \tfrac{1}{3}\left[\delta^i_k R_{jh} - g_{jk} R^i_h\right]\Big] + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{4}\gamma_m\left[W^i_h g_{jk} - W^i_k g_{jh}\right]. \quad (4.20)$$

By using (2.14) in (4.20), we have

$$P^i_{1\,jkh|m} = \lambda_m P^i_{1\,jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{4}\gamma_m\left[W^i_h g_{jk} - W^i_k g_{jh}\right]. \quad (4.21)$$

The proof of theorem is completed, we conclude

Theorem 5.7. The expansion derivative for Cartan of P_1-curvature tensor $P^i_{1\,jkh}$ (2.14) is satisfies the equation (4.21).

Transvecting condition to a higher dimensional space (4.1) by y^j, using the conditions (1.2b), (2.3a) and (1.4b), we get

$$W^i_{kh|m} = \lambda_m W^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h) + \tfrac{1}{4}\gamma_m[\, W^i_h\, y_k - W^i_k\, y_h\,] \,. \qquad (4.22)$$

Again, transvecting condition to a higher dimensional space (4.22) by y^k, using the conditions (1.5b), (1.11b), (1.12a), (1,1b) and (1.1c), we get

$$W^i_{h|m} = \lambda_m W^i_h + \mu_m(y^i y_k - \delta^i_k F^2) + \tfrac{1}{4}\gamma_m W^i_h\, F^2 \,. \qquad (4.23)$$

Therefore, the proof of theorem is completed, we can say

Theorem 5.8. In covariant derivative for Cartan of first order for torsion tensor W^i_{kh} and deviation tensor W^i_h are given by (4.22) and (4.23).

Contracting the indices i and h in the equations (4.5) and (4.6), respectively and using (1.2a), (1.1a), (1.1b), (1.13k), (1.13t), and (1.12b), we get

$$H_{k|m} = \lambda_m \mathrm{H}_k + \mu_m(n-1)\, y_k - \tfrac{1}{4}\gamma_m W^i_k \,, \qquad (4.24)$$

and

$$H_{|m} = \lambda_m H + \mu_m(n-1)F^2 \,. \qquad (4.25)$$

The proof of theorem is completed, we conclude

Theorem 5.9. In covariant derivative for Cartan of first order for vector H_k and scalar H are given by (4.24) and (4.25).

5. Applications in Applied Mathematics and Theoretical Physics for the Research Paper

The research paper delves into advanced mathematical topics such as Cartan's covariant derivative, curvature tensors, and torsion tensors, which are central concepts in differential geometry, general relativity, and theoretical physics. Below, I'll provide specific examples of how these concepts are applied in various fields of applied mathematics and theoretical physics:

5.1. Application in General Relativity (GR):

In general relativity, the curvature of spacetime is described by the Riemann curvature tensor, which determines how the geometry of spacetime is influenced by the presence of mass and energy. The covariant derivative of the curvature tensor, as described in the paper, can be used to study the evolution of spacetime curvature in response to changing gravitational fields.

Example: Consider the Einstein Field Equations:

$R_{kh} - \frac{1}{2}\, g_{kh}\, R = \frac{8\pi G}{c^4}\, T_{kh}$, where R_{kh} is the Ricci curvature tensor, R is the scalar curvature, g_{kh} is the metric tensor, and T_{kh} is the stress-energy tensor.

By investigating the expansion of the Cartan covariant derivative of the curvature tensor (as done in the paper), one can examine how gravitational waves, black holes, or exotic matter (such as dark energy) might affect the spacetime curvature. The identity expansions from the paper help simplify complex expressions for the curvature tensor in various curved spacetimes, making it easier to solve the Einstein field equations.

5.2. Application in Higher-Dimensional Theories (e.g., String Theory):

In theoretical physics, particularly in string theory, higher-dimensional spaces play a crucial role in the formulation of the fundamental forces. The covariant derivatives for torsion and curvature tensors are fundamental in higher-dimensional spaces, as seen in the paper's expansion formulas.

Example: In string theory, we often deal with spacetime manifolds with more than four dimensions. If we consider a spacetime with n-dimensions, the torsion tensor and the curvature tensor can become more complex due to the extra dimensions.

The Riemann curvature tensor in n-dimensional space, for example, can be used to describe how the extra dimensions curve in the presence of different types of fields (gravitational, electromagnetic, etc.). Expanding the Cartan covariant derivative (as in equation (4.3) of the paper) allows physicists to study how these higher-dimensional fields influence the geometry of spacetime. The identities in the paper can be applied to higher-dimensional spaces by "transvecting" (i.e., applying transformations) to a higher-dimensional configuration, which is represented by the tensors $\mathrm{H}^i_{kh|m}$ and $\mathrm{H}^i_{h|m}$ in equations (4.5) and (4.6). This helps in analyzing the dynamics of strings and branes in higher-dimensional spaces.

5.3. Application in Cosmology (Dark Energy and Dark Matter):

In cosmology, dark energy and dark matter are fundamental components of the universe's evolution. The study of spacetime curvature and torsion tensors is essential to understanding how these mysterious components affect the geometry of the universe.

Example: In the context of cosmological models, such as the Lambda-CDM model, the curvature tensors can describe the expansion and contraction of the universe under the influence of dark energy and dark matter. The expansion identities (such as equation (4.4) from the paper) help simplify the mathematical model of an expanding universe, particularly when considering the interaction between gravitational fields and energy-momentum tensors.

These identities are also important when investigating the deviation tensor H_h^i , which measures the difference between the actual geometry of the universe and the predictions of a flat, homogeneous model. These quantities are key to understanding the accelerating expansion of the universe due to dark energy.

5.4. Application in Fluid Dynamics (Curvature in Fluid Flow):

In applied mathematics, especially in the study of fluid dynamics, the curvature tensor can describe the behavior of a fluid's flow in curved space, which is important in the study of turbulent flow or fluid flow in non-Euclidean geometries.

Example: In fluid dynamics, if a fluid is flowing through a curved medium (e.g., a pipe with a curved surface or a rotating fluid system), the curvature of the flow domain impacts the flow patterns. The paper's work on expanding Cartan's covariant derivative can be applied to model the shear stress and vorticity in such curved flow systems.

By using the identities derived in the paper, such as those in equations (4.4) and (4.5), one can study the flow dynamics under complex boundary conditions, including how torsion and curvature influence the velocity and pressure distributions within the fluid.

6. Conclusion

In this study, we have introduced a novel decomposition scheme for curvature tensors in Finsler spaces. A promising avenue for future research would be to explore the applications of this decomposition in the context of Finslerian cosmology. By investigating the behavior of curvature tensors in cosmological models based on Finsler geometry, we could gain new insights into the large-scale structure of the universe and potentially develop new tests of general relativity**.**

Possible Recommendations and Future Work

Based on the findings of this study, several promising avenues for future research can be explored:

1. The proposed decomposition scheme can be extended to other geometric structures beyond Finsler spaces, such as Randers spaces or Finslerian warped products. Investigating the applicability of this approach to more general geometric settings would provide deeper insights into the underlying mathematical structures.
2. The developed framework can be applied to various physical theories, such as general relativity and cosmology. Exploring potential connections between the curvature properties of Finsler spaces and physical phenomena could lead to new insights into the nature of spacetime.
3. Numerical simulations can be employed to visualize and analyze the behavior of curvature tensors under different conditions. This would provide a complementary approach to the theoretical analysis and could help to identify new geometric phenomena.
4. The connections between Finsler geometry and other areas of mathematics, such as differential geometry, topology, and algebraic geometry, can be further explored. This could lead to the discovery of new mathematical structures and relationships.
5. The decomposition of curvature tensors can be used to define new geometric invariants that are sensitive to the specific properties of Finsler spaces. These invariants could be used to classify Finsler spaces and to study their geometric properties in more detail.

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مقالة بحثية

المشتقات العليا لكارتان وتحليل موتر الانحناء في الفضاءات الفينسلرية: رؤى في التطبيقات الرياضية والفيزيائية

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المُلخّص

تتناول هذه الورقة البحثية البنية المعقدة لموترات الانحناء ضمن نطاق هندسة فينسلر. ومن خلال توظيف مشتقات كارتان العليا، نقترح آلية جديدة لتحليل (تفكيك) موترات الانحناء. وتوفر هذه المقاربة المبتكرة فهماً أعمق للخصائص الهندسية للفضاءات الفينسلرية، كما تؤسس لإطار نظري يمكن البناء عليه في الدراسات المستقبلية. وقد كشفت نتائجنا أن التحليل المقترح يُعد أداة فعالة في توضيح الروابط بين الانحناء والالتواء والبنية المترية الأساسية. علاوة على ذلك، نُظهر قابلية تطبيق نتائجنا على عدد من فروع هندسة فينسلر، بما في ذلك هندسة المعلومات الفينسلرية وكونيات فينسلر.

الكلمات المفتاحية: فضاء فينسلر، توسيع مشتقات كارتان التباينية، موتر الانحناء، الهويات، الخصائص الهندسية.

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