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RESEARCH ARTICLE

ON THE (*p*, *q*)-ANALOGUE OF HERMITE MATRIX POLYNOMIALS IN THREE VARIABLES

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Abstract

The principal object of this paper is to study the (p, q)-analogue of Hermite matrix polynomials in three variables using the generating function method. This study shows a class of (p, q)-analogue of Hermite matrix polynomials with help of the generating functions such as explicit representation and some recurrence relations for these (p, q)-polynomials are derived. The construction and derivation of these results give us an idea of how to handle complex computation involving the parameters p and q.

Keywords: The (*p*, *q*)-Hermite matrix polynomials; Generating functions; (*p*, *q*)-calculus; Orthogonal polynomials; Recurrence relations.

1. Introduction and Preliminaries.

In the last quarter of 20th century, q-calculus appeared as a connection between mathematics and physics. We have also a generalization of q-calculus with one more parameter, we can say it is a two-parameter quantum calculus. Generally, it is called (p,q)-calculus. The theory (p,q)-calculus or post quantum calculus has recently been applied in many areas of mathematics, physics and engineering, such as biology, mechanics, economics, electrochemistry, probability theory, approximation theory, statistics, number theory, quantum theory, theory of relativity, and statistical mechanics, etc. for more details on this topic (p,q)-calculus, see, for example, [8,17,35]. Burban and Klimyk [35], Duran et al. [9,12,18,19], Jagannathan [33], Jagannathan and Srinivasa [25], Sahai and Yadav [24] have earlier investigated some properties of the two parameter quantum calculus. Sadjang [6,11,13] introduced the two (p,q)-analogues of the Laplace transform, two (p,q)-Taylor formulas for polynomials, (p,q)-Appell polynomials and developed some their properties. Mursaleen et al. [15,16] investigated the (p,q)-analogues of Bernstein operators and approximation properties of (p,q)- Bernstein operators that are a generalization of q- Bernstein operators. Khan and Lobiya [14] have nicely discussed a lot of applications in different approximation theory areas, such as per Weirstarass approximation theorems, basic hypergeometric functions, orthogonal polynomials and can be used in differential equations as well as computer-aided geometric designs.

In this section, we will give a summary of the mathematical notations and definitions required in this paper for the convenience of the reader.

Let the q-analogues of Pochhammer symbol or q-shifted factorial be defined by [3,4,11,13]

$$[\alpha]_q = \frac{1-q^\alpha}{1-q}, \quad 0<|q|<1; q \in C-\{1\}; \quad \alpha \in C \qquad (1.1)$$

Where

$$\lim_{q\to 1}[\alpha]_q = \lim_{q\to 1}\frac{1-q^\alpha}{1-q} = \alpha$$

The q-analogue of $n!$ is then defined by

$$[n]_q! = \begin{cases} 1, & n=0 \\ [n]_q[n-1]_q \dots [2]_q[1]_q, & n \in N \end{cases} \qquad (1.2)$$

Or

$$[n]_q! = \prod_{k=1}^{n}[k]_q, \quad [0]_q! = 1, \quad q \in C\backslash\{0,1\}$$

The (p,q)-number (bibasic number or twin-basic number) is denoted by $[\alpha]_{p,q}$ and is defined by the following notation

$$[\alpha]_{p,q} = \frac{p^\alpha - q^\alpha}{p-q}, \quad 0<|q|<|p|\le 1; \quad p,q,\alpha \in C. \qquad (1.3)$$

For $p,q,\alpha \in C$ and $0<|q|<|p|\le 1$, the (p,q)-number and (p,q)-factorial are given as follow. (see [3,4,6,17])

$$[n]_{p,q}=\begin{cases}\frac{p^n-q^n}{p-q}, & n\in N\\ 0, & n=0\end{cases}$$

The twin-basic number is a natural generalization of the q-number, that is

$$\lim_{p\to 1}[n]_{p,q}=[n]_q=\frac{1-q^n}{1-q}, q\neq 1$$

The (p,q)-factorial is defined by

$$[n]_{p,q}!=\prod_{k=1}^{n}[k]_{p,q},\ n\geq 1;\ \ [0]_{p,q}!=1. \qquad (1.4)$$

Let us introduce also the so-called (p,q)-binomial coefficient

$$\begin{bmatrix}n\\k\end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!},\ 0\leq k\leq n,\ k,n\in N. \qquad (1.5)$$

The (p,q)-exponential function is defined by (see [3,6])

$$e_{p,q}(x)=\sum_{k=0}^{\infty}p^{\binom{k}{2}}\frac{x^k}{[k]_{p,q}!}. \qquad (1.6)$$

The (p,q)-complementary exponential function is defined by

$$E_{p,q}(x)=\sum_{k=0}^{\infty}q^{\binom{k}{2}}\frac{x^k}{[k]_{p,q}!}. \qquad (1.7)$$

It is easy to see that (see [3,6])

$$e_{p,q}(x).E_{p,q}(-x)=1.$$

Let f be a function defined on a subset of real or complex plane. The (p,q)-derivative operator of the function f is defined as follows (see [6,13,24])

$$D_{p,q}f(x)=\frac{f(px)-f(qx)}{(p-q)x},\quad x\neq 0, \qquad (1.8)$$

and $((D_{p,q}f)(0)=f^{/}(0))$, provided that f is differentiable at 0, which satisfies the following relations (see [3,6])

$$D_{p,q}e_{p,q}(\mu x)=\mu e_{p,q}(\mu px),\ \mu\in C. \qquad (1.9)$$

$$D_{p,q}E_{p,q}(\mu x)=\mu E_{p,q}(\mu qx). \qquad (1.10)$$

It happens clearly that $D_{p,q}x^n=[n]_{p,q}x^{n-1}$. Note also that for $p=1$, the (p, q)-derivative reduces to the Hahn derivative given by

$$D_qf(x)=\frac{f(x)-f(qx)}{(1-q)x},\quad x\neq 0$$

The (p,q)-derivative operator satisfy the following product rules as follows: (see [3,611,13])

$$D_{p,q}(f(x).g(x))=f(px)D_{p,q}g(x)+g(qx)D_{p,q}f(x), \qquad (1.11)$$

Also, the q-analogue of $(x\pm y)_q^n$ is given by [21]

$$(x\pm y)_q^n=(x\pm y)_n=x^n(\mp {}^{y}/_{x};q)=x^n\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{k}{2}}(\mp {}^{y}/_{x})^k. \qquad (1.12)$$

Let us introduce also the so-called the (p,q)-powers [2,11,23]

$$\left.\begin{aligned}(x\ominus a)_{p,q}^n&=(x-a)(px-qa)\dots(xp^{n-1}-aq^{n-1})\\(x\oplus a)_{p,q}^n&=(x+a)(px+qa)\dots(xp^{n-1}+aq^{n-1})\end{aligned}\right\}, \qquad (1,13)$$

$$(x\oplus y)_{p,q}^n=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}q^{\binom{k}{2}}x^ky^{n-k}. \qquad (1.14)$$

Hermite Polynomials are defined by means of generating relations [39]

$$exp[2xt-t^2]=\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}, \qquad (1.15)$$

$$exp[2xt+yt^2]=\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!}. \qquad (1.16)$$

Shrivastava [10] presented and studied the classical Hermite polynomials and its generalizations in the form:

$$exp[2x(t+h)-(y+1)(t+h)^2]=\sum_{n,m=0}^{\infty}H_{n,m}(x,y)\frac{t^nh^m}{n!m!}. \qquad (1.17)$$

Jodar and Company [34] introduced the class of Hermite matrix polynomials $H_n(x,A)$ defined by

$$exp[xt\sqrt{2A}-t^2I]=\sum_{n=0}^{\infty}H_n(x,A)\frac{t^n}{n!}, \qquad (1.18)$$

and

$$H_n(x,A)=n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k}{k!(n-2k)!}\left(x\sqrt{2A}\right)^{n-2k};\quad n\geq 0. \qquad (1.19)$$

which appear a finite series solutions of second order matrix differential equations $y^{//}-xAy^{/}+nAy=0$, for a matrix A in $C^{N\times N}$ whose eigenvalues are all in the right open half-plane.

In [27] Sayyed, Metwally and Batahan introduced a generalization of the Hermite matrix polynomials of the form

$$F(x,t)=exp[\lambda(xt\sqrt{2A}-t^2I)]=\sum_{n=0}^{\infty}H_{n,m}^{\lambda}(x,A)\frac{t^n}{n!}. \qquad (1.20)$$

Also, Batahan [26] presented a study of the two-variable Hermite matrix polynomials defined by

$$F(x,y,t)=exp[xt\sqrt{2A}-yt^2I]=\sum_{n=0}^{\infty}H_n(x,y,A)\frac{t^n}{n!}, \qquad (1.21)$$

where

$$H_n(x,y,A)=n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k}{k!(n-2k)!}\left(x\sqrt{2A}\right)^{n-2k}y^k, \qquad (1.22)$$

Moreover, Kahmmash [22] introduced and studied the Hermite matrix polynomials of two variables defined by

$$exp[xt\sqrt{2A}-(y+1)t^2I]=\sum_{n=0}^{\infty}H_n(x,y,A)\frac{t^n}{n!},\quad |t^n|<\infty \qquad (1.23)$$

where

$$H_n(x,y,A)=n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k}{k!(n-2k)!}\left(x\sqrt{2A}\right)^{n-2k}(y+1)^k. \qquad (1.24)$$

Pathan, Bin Saad and Alsarahi [20] studied on matrix polynomials associated with Hermite matrix polynomials

$$exp[x(t+h)\sqrt{2A}-y(t+h)^2I]=\sum_{n,m=0}^{\infty}H_{n,m}(x,y;A)\frac{t^nh^m}{n!m!}. \qquad (1.25)$$

Also, Alsarahi [1] presented a study of the generalized q-analogue Hermite matrix polynomials of two variables

$$H_{n,m}(x,y,a;A;q) = \sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{m}{2}\right]}(-1)^{(r+s)(a+1)}\frac{q^{\frac{a}{4}(n+m-2r-2s)^2+\frac{a}{4}(r+s)^2+\binom{m-2s}{2}+\binom{2s}{2}}(q;q)_{2r+2s}}{(q;q)_{n-2r}(q;q)_{m-2s}(q;q)_{r+s}(q;q)_{2r}(q;q)_{2s}} \times \left(x\sqrt{2A}\right)^{n+m-2r-2s} y^{r+s}. \quad (1.26)$$

The following double series transformations that we will occasionally use, are easy to prove [39]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}A(k,n-2k), \quad (1.27)$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k). \quad (1.28)$$

Similarly, we can write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+2k), \quad (1.29)$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}A(k,n-k), \quad (1.30)$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+k), \quad (1.31)$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+mk), \quad (1.32)$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}A(k,n-(m-1)k), \quad (1.33)$$

where m is a positive integer and $n > m$.

The main purpose of this paper is to obtain explicit formula of (p,q)-analogue of Hermite matrix polynomial in three variables for $0 < |q| < |p| \leq 1$, $q,p \in C$. We mainly use the (p,q)-calculus in the theory of special functions. This work is organized as follows. More precisely, we define the numerous (known or new) (p,q)-Hermite matrix polynomial of three variables and discuss some significant properties such as explicit representations and some interesting differential recurrence relations for the (p,q)-Hermite matrix polynomial of three variables are discussed.

2. The (p,q)-Hermite Matrix Polynomials of Three Variables

We introduce the (p,q)-Hermite matrix polynomial of three variables by the following:

$$H^{k}_{n,m,u}(x,y,z;A;p,q) = \sum_{r=0}^{\left[\frac{n}{k}\right]}\sum_{s=0}^{\left[\frac{m}{k}\right]}\sum_{v=0}^{\left[\frac{u}{k}\right]}(-1)^{r+s+v} \times \frac{p^{\binom{n-kr+m-ks+u-kv}{2}+\binom{n-kr}{2}+\binom{m-ks}{2}+\binom{r+s+v}{2}+\binom{kr}{2}+\binom{ks}{2}}q^{\binom{m-ks+u-kv}{2}+\binom{u-kv}{2}+\binom{ks}{2}+\binom{kv}{2}}}{[n-kr]_{p,q}![m-ks]_{p,q}![u-kv]_{p,q}![r+s+v]_{p,q}![kr]_{p,q}![ks]_{p,q}![kv]_{p,q}!} \times [kr+ks+kv]_{p,q}!\left(x\sqrt{2A}\right)^{n-kr+m-ks+u-kv}(y+z)^{r+s+v}. \quad (2.1)$$

Now, we get generating function of the (p,q)-analogue Hermite matrix polynomials in the form of the following theorem:

***Theorem* 2.1.** Let A be a positive stable matrix in $C^{N\times N}$ and $0 < |q| < |p| \leq 1$, $q,p \in C$, the following generating function for the (p,q)-analogue Hermite matrix polynomials $H^{k}_{n,m,u}(x,y,z;A;p,q)$ holds true:

$$\sum_{n,m,u=0}^{\infty}H^{k}_{n,m,u}(x,y,z;A;p,q)\,t^n h^m g^u = exp_{p,q}\left(x\sqrt{2A}(g+h+t)-(y+z)(g+h+t)^k I\right). \quad (2.2)$$

***Proof*.** Let us denote the right hand side of (2.2) by W, then

$$W = e_{p,q}\left(x\sqrt{2A}(g+h+t)\right).e_{p,q}\left(-(y+z)(g+h+t)^k\right)$$

Appling relation (1.6), we obtain

$$W = \sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}\left(x\sqrt{2A}\right)^n}{[n]_{p,q}!}(g+h+t)^n_{p,q}\sum_{r=0}^{\infty}(-1)^r\frac{p^{\binom{r}{2}}(y+z)^r}{[r]_{p,q}!}(g+h+t)^{kr}_{p,q}, \quad (2.3)$$

which using relation (1.14), we find

$$W = \sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}\left(x\sqrt{2A}\right)^n}{[n]_{p,q}!}\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}_{p,q}p^{\binom{n-m}{2}}q^{\binom{m}{2}}t^{n-m}(g+h)^m_{p,q} \times \sum_{r=0}^{\infty}(-1)^r\frac{p^{\binom{r}{2}}(y+z)^r}{[r]_{p,q}!}\sum_{s=0}^{kr}\begin{bmatrix}kr\\s\end{bmatrix}_{p,q}p^{\binom{kr-s}{2}}q^{\binom{s}{2}}t^{kr-s}(g+h)^s_{p,q}, \quad (2.4)$$

Using relations (1.5) and again (1.14), we obtain

$$W = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{u=0}^{m}\frac{p^{\binom{n}{2}+\binom{n-m}{2}+\binom{m-u}{2}}q^{\binom{m}{2}+\binom{u}{2}}\left(x\sqrt{2A}\right)^n}{[n-m]_{p,q}![m-u]_{p,q}![u]_{p,q}!}t^{n-m}h^{m-u}g^u \times \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{v=0}^{s}(-1)^r\frac{p^{\binom{r}{2}+\binom{kr-ks}{2}+\binom{ks-kv}{2}}q^{\binom{ks}{2}+\binom{kv}{2}}(y+z)^r[kr]_{p,q}!}{[r]_{p,q}![kr-ks]_{p,q}![ks-kv]_{p,q}![kv]_{p,q}!}t^{kr-ks}h^{ks-kv}g^{kv}, \quad (2.5)$$

by using relation (1.31) in (2.5), we get

$$W = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{u=0}^{\infty}\frac{p^{\binom{n+m+u}{2}+\binom{n}{2}+\binom{m}{2}}q^{\binom{m+u}{2}+\binom{u}{2}}\left(x\sqrt{2A}\right)^{n+m+u}}{[n]_{p,q}![m]_{p,q}![u]_{p,q}!}t^n h^m g^u \times \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{v=0}^{\infty}(-1)^{r+s+v}\frac{p^{\binom{r+s+v}{2}+\binom{kr}{2}+\binom{ks}{2}}q^{\binom{ks}{2}+\binom{kv}{2}}(y+z)^{r+s+v}}{[r+s+v]_{p,q}![kr]_{p,q}![ks]_{p,q}![kv]_{p,q}!}[kr+ks+kv]_{p,q}!\,t^{kr}h^{ks}g^{ku}, \quad (2.6)$$

thus, by using relation (1.27) in (2.6), we find

$$W = \sum_{n,m,u=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{k}\right]}\sum_{s=0}^{\left[\frac{m}{k}\right]}\sum_{v=0}^{\left[\frac{u}{k}\right]}(-1)^{r+s+v} \times \frac{p^{\binom{n-kr+m-ks+u-kv}{2}+\binom{n-kr}{2}+\binom{m-ks}{2}+\binom{r+s+v}{2}+\binom{kr}{2}+\binom{ks}{2}}q^{\binom{m-ks+u-kv}{2}+\binom{u-kv}{2}+\binom{ks}{2}+\binom{kv}{2}}}{[n-kr]_{p,q}![m-ks]_{p,q}![u-kv]_{p,q}![r+s+v]_{p,q}![kr]_{p,q}![ks]_{p,q}![kv]_{p,q}!} \times [kr+ks+kv]_{p,q}!\left(x\sqrt{2A}\right)^{n-kr+m-ks+u-kv}(y+z)^{r+s+v}t^n h^m g^u.$$

By using definition (2.2), we obtain the required relation (2.1).

***Lemma* 2.1.** The polynomial $H^{k}_{n,m,u}(x,y,z;A;p,q)$ is a (p,q)-analogy of each of the Hermite matrix polynomials and the modified Hermite matrix polynomials.

Proof.

$$\lim_{p\to 1}H^{k}_{n,m,u}(x,y,z;A;p,q) = \lim_{p\to 1}\sum_{r=0}^{\left[\frac{n}{k}\right]}\sum_{s=0}^{\left[\frac{m}{k}\right]}\sum_{v=0}^{\left[\frac{u}{k}\right]}(-1)^{r+s+v} \times \frac{p^{\binom{n-kr+m-ks+u-kv}{2}+\binom{n-kr}{2}+\binom{m-ks}{2}+\binom{r+s+v}{2}+\binom{kr}{2}+\binom{ks}{2}}q^{\binom{m-ks+u-kv}{2}+\binom{u-kv}{2}+\binom{ks}{2}+\binom{kv}{2}}}{[n-kr]_{p,q}![m-ks]_{p,q}![u-kv]_{p,q}![r+s+v]_{p,q}![kr]_{p,q}![ks]_{p,q}![kv]_{p,q}!} \times [kr+ks+kv]_{p,q}!\left(x\sqrt{2A}\right)^{n-kr+m-ks+u-kv}y^{r+s+v}$$

$$= \sum_{r=0}^{\left[\frac{n}{k}\right]} \sum_{s=0}^{\left[\frac{m}{k}\right]} \sum_{v=0}^{\left[\frac{u}{k}\right]} (-1)^{r+s+v} \frac{q^{\binom{m-ks+u-kv}{2}+\binom{u-kv}{2}+\binom{ks}{2}+\binom{kv}{2}} [kr+ks+kv]_q!}{[n-kr]_q! [m-ks]_q! [u-kv]_q! [r+s+v]_q!}$$

$$\frac{\left(x\sqrt{2A}\right)^{n-kr+m-ks+u-kv} y^{r+s+v}}{[kr]_q! \, [ks]_q! \, [kv]_q!}$$

Hence, we get

$$\lim_{p\to 1} H^k_{n,m,u}(x,y,z;A;p,q) = H^k_{n,m}(x,y,z;A;q).$$

Thus

$$\lim_{\substack{p\to 1\\ q\to 1}} H^k_{n,m,u}(x,y,0;A;p,q) = H^k_{n,m}(x,y;A). \qquad (2.7)$$

Putting $k = 2$, and $z = 0$ in (2.7), we obtain the known result (1.25).

Also $k = 2$ and replacing y by $y + 1$ in (2.7), we obtain the result (1.23).

Also, we define that the generating function of (p, q)-Hermite matrix polynomials $H^k_{n,m,u}(x, y, z; A; p, q)$ of the second form in the following theorem:

***Theorem* 2.2.** Let A be a positive stable matrix in $C^{N\times N}$ and $0 < |q| < |p| \le 1,\ q, p \in C,$ the following generating function for the (p, q)-Hermite matrix polynomials $H^k_{n,m,u}(x, y, z; A; p, q)$ holds true:

$$\sum_{n,m,u=0}^{\infty} H^k_{n,m,u}(x,y,z;A;p,q)\, t^n h^m g^u$$

$$= E_{p,q}\left(x\sqrt{2A}(g+h+t) - (y+z)(g+h+t)^k I\right) . \qquad (2.8)$$

***Proof*.** Let us denote the right hand side of (2.2) by U, then

$$U = E_{p,q}\left(x\sqrt{2A}(g+h+t)\right) . E_{p,q}(-(y+z)(g+h+t)^k)$$

Appling relation (1.7), we obtain

$$U = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (x\sqrt{2A})^n}{[n]_{p,q}!} (g+h+t)^n_{p,q} \sum_{r=0}^{\infty} (-1)^r \frac{q^{\binom{r}{2}} (y+z)^r}{[r]_{p,q}!} (g+h+t)^{kr}_{p,q},$$
(2.9)

from relation (1.14), we find

$$U = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} \left(x\sqrt{2A}\right)^n}{[n]_{p,q}!} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_{p,q} p^{\binom{n-m}{2}} q^{\binom{m}{2}} t^{n-m} (g+h)^m_{p,q}$$

$$\times \sum_{r=0}^{\infty} (-1)^r \frac{q^{\binom{r}{2}} (y+z)^r}{[r]_{p,q}!} \sum_{s=0}^{kr} \begin{bmatrix} kr \\ s \end{bmatrix}_{p,q} p^{\binom{kr-s}{2}} q^{\binom{s}{2}} t^{kr-s} (g+h)^s_{p,q},$$
(2.10)

which using relations (1.5) and again (1.14), we obtain

$$U = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{u=0}^{m} \frac{p^{\binom{n-m}{2}+\binom{m-u}{2}} q^{\binom{n}{2}+\binom{m}{2}+\binom{u}{2}} \left(x\sqrt{2A}\right)^n}{[n-m]_{p,q}! \, [m-u]_{p,q}! \, [u]_{p,q}!} t^{n-m} h^{m-u} g^u$$

$$\times \sum_{r=0}^{\infty} \sum_{s=0}^{r} \sum_{v=0}^{s} (-1)^r \frac{p^{\binom{kr-ks}{2}+\binom{ks-kv}{2}} q^{\binom{r}{2}+\binom{ks}{2}+\binom{kv}{2}} (y+z)^r [kr]_{p,q}!}{[r]_{p,q}! [kr-ks]_{p,q}! [ks-kv]_{p,q}! [kv]_{p,q}!} t^{kr-ks} h^{ks-kv} g^{kv},$$
(2.11)

by using relation (1.31) in (2.11), we get

$$U = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{u=0}^{\infty} \frac{p^{\binom{n}{2}+\binom{m}{2}} q^{\binom{n+m+u}{2}+\binom{m+u}{2}+\binom{u}{2}} \left(x\sqrt{2A}\right)^{n+m+u}}{[n]_{p,q}! \, [m]_{p,q}! \, [u]_{p,q}!} t^n h^m g^u$$

$$\times \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{v=0}^{\infty} (-1)^{r+s+v} \frac{p^{\binom{kr}{2}+\binom{ks}{2}} q^{\binom{r+s+v}{2}+\binom{ks}{2}+\binom{kv}{2}} (y+z)^{r+s+v}}{[r+s+v]_{p,q}! \, [kr]_{p,q}! \, [ks]_{p,q}! \, [kv]_{p,q}!}$$

$$\times [kr+ks+kv]_{p,q}! \, t^{kr} h^{ks} g^{ku}, \qquad (2.12)$$

also, by using relation (1.27) in (2.12), we find

$$U = \sum_{n,m,u=0}^{\infty} \sum_{r=0}^{\left[\frac{n}{k}\right]} \sum_{s=0}^{\left[\frac{m}{k}\right]} \sum_{v=0}^{\left[\frac{u}{k}\right]} (-1)^{r+s+v}$$

$$\times \frac{p^{\binom{n-kr}{2}+\binom{m-ks}{2}+\binom{kr}{2}+\binom{ks}{2}} q^{\binom{n-kr+m-ks+u-kv}{2}+\binom{m-ks+u-kv}{2}+\binom{u-kv}{2}+\binom{r+s+v}{2}+\binom{ks}{2}+\binom{kv}{2}}}{[n-kr]_{p,q}! \, [m-ks]_{p,q}! \, [u-kv]_{p,q}! \, [r+s+v]_{p,q}! \, [kr]_{p,q}! \, [ks]_{p,q}! \, [kv]_{p,q}!}$$

$$\times [kr+ks+kv]_{p,q}! \left(x\sqrt{2A}\right)^{n-kr+m-ks+u-kv} (y+z)^{r+s+v} t^n h^m g^u.$$

By equating the coefficients of $t^n h^m g^u$, we obtain the other relation

$$H^k_{n,m,u}(x,y,z;A;p,q) = \sum_{r=0}^{\left[\frac{n}{k}\right]} \sum_{s=0}^{\left[\frac{m}{k}\right]} \sum_{v=0}^{\left[\frac{u}{k}\right]} (-1)^{r+s+v}$$

$$\times \frac{p^{\binom{n-kr}{2}+\binom{m-ks}{2}+\binom{kr}{2}+\binom{ks}{2}} q^{\binom{n-kr+m-ks+u-kv}{2}+\binom{m-ks+u-kv}{2}+\binom{u-kv}{2}+\binom{r+s+v}{2}+\binom{ks}{2}+\binom{kv}{2}}}{[n-kr]_{p,q}! \, [m-ks]_{p,q}! \, [u-kv]_{p,q}! \, [r+s+v]_{p,q}! \, [kr]_{p,q}! \, [ks]_{p,q}! \, [kv]_{p,q}!}$$

$$\times [kr+ks+kv]_{p,q}! \left(x\sqrt{2A}\right)^{n-kr+m-ks+u-kv} (y+z)^{r+s+v}, \qquad (2.13)$$

which is other representation of $H^k_{n,m,u}(x, y, z; A; p, q)$.

3. Recurrence Relations

Recurrence relations are central to both theoretical analysis and numerical computation of special functions. For the (p, q)-Hermite matrix polynomials of three-index and three-variable, the following recurrence relations hold.

***Theorem* 3.1.** The (p, q)-Hermite matrix polynomials of three-index and three-variable $H^k_{n,m,u}(x, y, z; A; p, q)$ satisfy the following relations:

$$\frac{\partial^s_{p,q}}{\partial_{p,q} x^s} H^k_{n,m,u}(x,y,z;A;p,q)$$

$$= p^{s-1}\left(\sqrt{2A}\right)^s \sum_{r=0}^{s} \sum_{v=0}^{r} \frac{p^{\binom{r}{2}} q^{\binom{s}{2}} [s]_{p,q}!}{[s-r]_{p,q}! \, [r-v]_{p,q}! \, [v]_{p,q}!}$$

$$\times H^k_{n+r-s,m+v-r,u-v}(p^s x, y, z; A; p, q), \qquad (3.1)$$

$$\frac{\partial^s_{p,q}}{\partial_{p,q} y^s} H^k_{n,m,u}(x,y,z;A;p,q)$$

$$= (-1)^s p^{s-1} \sum_{r=0}^{sk} \sum_{v=0}^{r} \frac{p^{\binom{sk-r}{2}+\binom{r-v}{2}} q^{\binom{r}{2}+\binom{v}{2}} [sk]_{p,q}!}{[sk-r]_{p,q}! \, [r-v]_{p,q}! \, [v]_{p,q}!}$$

$$\times H^k_{n+r-sk,m+v-r,u-v}(x, p^s y, p^s z; A; p, q), \qquad (3.2)$$

and

$$\frac{\partial^s_{p,q}}{\partial_{p,q} z^s} H^k_{n,m,u}(x,y,z;A;p,q)$$

$$= (-1)^s p^{s-1} \sum_{r=0}^{sk} \sum_{v=0}^{r} \frac{p^{\binom{sk-r}{2}+\binom{r-v}{2}} q^{\binom{r}{2}+\binom{v}{2}} [sk]_{p,q}!}{[sk-r]_{p,q}! \, [r-v]_{p,q}! \, [v]_{p,q}!}$$

$$\times H^k_{n+r-sk,m+v-r,u-v}(x, p^s y, p^s z; A; p, q), \qquad (3.3)$$

***Proof*.** Differentiating (2.2) with respect to x yields

$$\sum_{n,m,u=0}^{\infty} \frac{\partial_{p,q}}{\partial_{p,q} x} H^k_{n,m,u}(x,y,z;A;p,q)\, t^n h^m g^u$$

$$= \sqrt{2A}(g+h+t)_{p,q} e_{p,q}\left(px\sqrt{2A}(g+h+t)\right) . e_{p,q}(-(y+z)(g+h+t)^k)$$

$$= \sqrt{2A}\sum_{r=0}^{1}\frac{[1]_{p,q}!\,t^{1-r}(g+h)_{p,q}^{r}}{[1-r]_{p,q}!\,[r]_{p,q}!}\sum_{n,m,u=0}^{\infty}H_{n,m,u}^{k}(px,y,z;A;p,q)\,t^n h^m g^u,$$

which on using relations (1.14) and (1.5), gives

$$\sum_{n,m,u=0}^{\infty}\frac{\partial_{p,q}}{\partial_{p,q}x}H_{n,m,u}^{k}(x,y,z;A;p,q)\,t^n h^m g^u$$
$$= \sqrt{2A}\sum_{r=0}^{1}\sum_{v=0}^{r}\frac{p^{\binom{r}{2}}[1]_{p,q}!}{[1-r]_{p,q}!\,[r-v]_{p,q}!\,[v]_{p,q}!}$$
$$\times \sum_{n,m,u=0}^{\infty}H_{n,m,u}^{k}(px,y,z;A;p,q)t^{n+1-r}h^{m+r-v}g^{u+v}$$
$$= \sqrt{2A}\sum_{n,m=0}^{\infty}\sum_{r=0}^{1}\sum_{v=0}^{r}\frac{p^{\binom{r}{2}}[1]_{p,q}!}{[1-r]_{p,q}![r-v]_{p,q}![v]_{p,q}!}H_{n+r-1,m+v-r,u-v}^{k}(px,y,z;A;p,q)t^n h^m g^u$$

On comparing the coefficients of $t^n h^m g^u$ on both sides of the above equation, we obtain

$$\frac{\partial_{p,q}}{\partial_{p,q}x}H_{n,m,u}^{k}(x,y,z;A;p,q)$$
$$= \sqrt{2A}\sum_{r=0}^{1}\sum_{v=0}^{r}\frac{p^{\binom{r}{2}}[1]_{p,q}!}{[1-r]_{p,q}!\,[r-v]_{p,q}!\,[v]_{p,q}!}H_{n+r-1,m+v-r,u-v}^{k}(px,y,z;A;p,q)$$

Thus

$$\frac{\partial_{p,q}^{2}}{\partial_{p,q}x^{2}}H_{n,m,u}^{k}(x,y,z;A;p,q)$$
$$= p\left(\sqrt{2A}\right)^{2}\sum_{r=0}^{2}\sum_{v=0}^{r}\frac{p^{\binom{r}{2}}[2]_{p,q}!}{[2-r]_{p,q}!\,[r-v]_{p,q}!\,[v]_{p,q}!}H_{n+r-2,m+v-r,u-v}^{k}(p^2x,y,z;A;p,q)$$

Hence

$$\frac{\partial_{p,q}^{s}}{\partial_{p,q}x^{s}}H_{n,m,u}^{k}(x,y,z;A;p,q) =$$
$$p^{s-1}\left(\sqrt{2A}\right)^{s}\sum_{r=0}^{s}\sum_{v=0}^{r}\frac{p^{\binom{r}{2}}q^{\binom{s}{2}}[s]_{p,q}!}{[s-r]_{p,q}![r-v]_{p,q}![v]_{p,q}!}H_{n+r-s,m+v-r,u-v}^{k}(p^s x,y,z;A;p,q)$$

which is the required relation (3.1).

Similarly, differentiating (2.2) with respect to y yields

$$\sum_{n,m,u=0}^{\infty}\frac{\partial_{p,q}}{\partial_{p,q}y}H_{n,m,u}^{k}(x,y,z;A;p,q)\,t^n h^m g^u$$
$$= -(g+h+t)_{p,q}^{k}.\,e_{p,q}\left(x\sqrt{2A}(g+h+t)\right).\,e_{p,q}(-p(y+z)(g+h+t)^k)$$
$$= -\sum_{r=0}^{k}\begin{bmatrix}k\\r\end{bmatrix}_{p,q}p^{\binom{k-r}{2}}q^{\binom{r}{2}}\,t^{k-r}(g$$
$$+h)_{p,q}^{r}\sum_{n,m,u=0}^{\infty}H_{n,m,u}^{k}(x,py,pz;A;p,q)\,t^n h^m g^u$$

By using relations (1.14) and (1.5), we find

$$\sum_{n,m,u=0}^{\infty}\frac{\partial_{p,q}}{\partial_{p,q}y}H_{n,m,u}^{k}(x,y,z;A;p,q)\,t^n h^m g^u$$
$$= -\sum_{r=0}^{k}\sum_{v=0}^{r}\frac{p^{\binom{k-r}{2}+\binom{r-v}{2}}q^{\binom{r}{2}+\binom{v}{2}}[k]_{p,q}!}{[k-r]_{p,q}!\,[r-v]_{p,q}!\,[v]_{p,q}!}$$
$$\times \sum_{n,m,u=0}^{\infty}H_{n,m,u}^{k}(x,py,pz;A;p,q)\,t^{n+k-r}h^{m+r-v}g^{u+v}$$
$$= -\sum_{n,m,u=0}^{\infty}\sum_{r=0}^{k}\sum_{v=0}^{r}\frac{p^{\binom{k-r}{2}+\binom{r-v}{2}}q^{\binom{r}{2}+\binom{v}{2}}[k]_{p,q}!}{[k-r]_{p,q}![r-v]_{p,q}![v]_{p,q}!}H_{n+r-k,m+v-r,u-v}^{k}(x,py,pz;A;p,q)t^n h^m g^u$$

By equating the coefficients of $t^n h^m g^u$ on both sides of the above equation, we get

$$\frac{\partial_{p,q}}{\partial_{p,q}y}H_{n,m,u}^{k}(x,y,z;A;p,q)$$
$$= -\sum_{r=0}^{k}\sum_{v=0}^{r}\frac{p^{\binom{k-r}{2}+\binom{r-v}{2}}q^{\binom{r}{2}+\binom{v}{2}}[k]_{p,q}!}{[k-r]_{p,q}!\,[r-v]_{p,q}!\,[v]_{p,q}!}H_{n+r-k,m+v-r,u-v}^{k}(x,py,pz;A;p,q)$$

Thus

$$\frac{\partial_{p,q}^{2}}{\partial_{p,q}y^{2}}H_{n,m,u}^{k}(x,y,z;A;p,q)$$
$$= (-1)^2 p\sum_{r=0}^{2k}\sum_{v=0}^{r}\frac{p^{\binom{2k-r}{2}+\binom{r-v}{2}}q^{\binom{r}{2}+\binom{v}{2}}[2k]_{p,q}!}{[2k-r]_{p,q}!\,[r-v]_{p,q}!\,[v]_{p,q}!}H_{n+r-2k,m+v-r,u-v}^{k}(x,p^2y,p^2z;A;p,q)$$

Hence

$$\frac{\partial_{p,q}^{s}}{\partial_{p,q}y^{s}}H_{n,m,u}^{k}(x,y,z;A;p,q)$$
$$= (-1)^s p^{s-1}\sum_{r=0}^{sk}\sum_{v=0}^{r}\frac{p^{\binom{sk-r}{2}+\binom{r-v}{2}}q^{\binom{r}{2}+\binom{v}{2}}[sk]_{p,q}!}{[sk-r]_{p,q}!\,[r-v]_{p,q}!\,[v]_{p,q}!}H_{n+r-sk,m+v-r,u-v}^{k}(x,p^s y,p^s z;A;p,q)$$

which is the required relation (3.2).

In same way, we get the relation (3.3).

***Theorem* 3.2.** The polynomials sequence $H_{n,m,u}^{k}(x,y,z;A;p,q)$ satisfies the next recurrence relations

$$[n+1]_{p,q}H_{n+1,m,u}^{k}(x,y,z;A;p,q) = \sqrt{2A}xH_{n,m,u}^{k}(px,qy,qz;A;p,q)$$
$$-(y+z)\sum_{r=0}^{k-1}\sum_{v=0}^{r}\frac{p^{\binom{k-1-r}{2}+\binom{r-v}{2}}q^{\binom{r}{2}+\binom{v}{2}}[k]_{p,q}!}{[k-r-1]_{p,q}![r-v]_{p,q}![v]_{p,q}!}H_{n+r+1-k,m+v-r,u-v}^{k}(px,py,pz;A;p,q),$$

(3.4)

$$[m+1]_{p,q}H_{n,m+1,u}^{k}(x,y,z;A;p,q) = \sqrt{2A}xH_{n,m,u}^{k}(px,qy,qz;A;p,q)$$
$$-(y+z)\sum_{r=0}^{k-1}\sum_{v=0}^{r}\frac{p^{\binom{k-1-r}{2}+\binom{r-v}{2}}q^{\binom{r}{2}+\binom{v}{2}}[k]_{p,q}!}{[k-r-1]_{p,q}![r-v]_{p,q}![v]_{p,q}!}H_{n+r+1-k,m+v-r,u-v}^{k}(px,py,pz;A;p,q),$$

(3.5)

and,

$$[u+1]_{p,q}H_{n,m,u+1}^{k}(x,y,z;A;p,q) = \sqrt{2A}xH_{n,m,u}^{k}(px,qy,qz;A;p,q)$$
$$-(y+z)\sum_{r=0}^{k-1}\sum_{v=0}^{r}\frac{p^{\binom{k-1-r}{2}+\binom{r-v}{2}}q^{\binom{r}{2}+\binom{v}{2}}[k]_{p,q}!}{[k-r-1]_{p,q}![r-v]_{p,q}![v]_{p,q}!}H_{n+r+1-k,m+v-r,u-v}^{k}(px,py,pz;A;p,q).$$

(3.6)

***Proof*.** Differentiating (2.2) with respect to t and using (1.11), we find

$$\sum_{n,m=0}^{\infty}\frac{\partial_{p,q}}{\partial_{p,q}t}H_{n,m,u}^{k}(x,y,z;A;p,q)\,t^n h^m g^u$$
$$= \sqrt{2A}xe_{p,q}\left(px\sqrt{2A}(g+h+t)\right).\,e_{p,q}(-q(y+z)(g+h+t)^k)$$
$$-[k]_{p,q}(y+z)(g+h+t)_{p,q}^{k-1}e_{p,q}\left(px\sqrt{2A}(g+h+t)\right)e_{p,q}(-p(y+z)(g+h+t)^k)$$

By using relation (1.14) and definition (2.2), we get

$$\sum_{n,m=0}^{\infty}[n]_{p,q}H_{n,m,u}^{k}(x,y,z;A;p,q)\,t^{n-1}h^m g^u$$
$$= \sqrt{2A}x\sum_{n,m,u=0}^{\infty}H_{n,m,u}^{k}(px,qy,qz;A;p,q)\,t^n h^m g^u$$

$$-[k]_{p,q}(y+z)\sum_{r=0}^{k-1}\begin{bmatrix}k-1\\ r\end{bmatrix}_{p,q} p^{\binom{k-1}{2}}\, t^{k-r-1}(g+h)_{p,q}^{r}\sum_{n,m,u=0}^{\infty} H_{n,m,u}^{k}(px,py,pz;A;p,q)\, t^n h^m g^u$$

Appling relation (1.5), we obtain

$$\sum_{n,m=0}^{\infty}[n+1]_{p,q}H_{n+1,m,u}^{k}(x,y,z;A;p,q)\, t^n h^m g^u$$

$$=\sqrt{2A}x\sum_{n,m,u=0}^{\infty}H_{n,m,u}^{k}(px,qy,qz;A;p,q)\, t^n h^m g^u$$

$$-[k]_{p,q}(y+z)\sum_{r=0}^{k-1}\sum_{v=0}^{r}\frac{p^{\binom{k-1-r}{2}+\binom{r-v}{2}}q^{\binom{r}{2}+\binom{v}{2}}[k-1]_{p,q}!}{[k-r-1]_{p,q}!\,[r-v]_{p,q}!\,[v]_{p,q}!}$$

$$\times\sum_{n,m,u=0}^{\infty}H_{n,m,u}^{k}(px,py,pz;A;p,q)\, t^{n+k-r-1}h^{m+r-v}g^{u+v}$$

thus

$$\sum_{n,m=0}^{\infty}[n+1]_{p,q}H_{n+1,m,u}^{k}(x,y,z;A;p,q)\, t^n h^m g^u$$

$$=\sqrt{2A}x\sum_{n,m,u=0}^{\infty}H_{n,m,u}^{k}(px,qy,qz;A;p,q)\, t^n h^m g^u$$

$$-[k]_{p,q}(y+z)\sum_{r=0}^{k-1}\sum_{v=0}^{r}\frac{p^{\binom{k-1-r}{2}+\binom{r-v}{2}}q^{\binom{r}{2}+\binom{v}{2}}[k-1]_{p,q}!}{[k-r-1]_{p,q}!\,[r-v]_{p,q}!\,[v]_{p,q}!}$$

$$\times\sum_{n,m,u=0}^{\infty}H_{n+r+1-k,m+v-r,u-v}^{k}(px,py,pz;A;p,q)\, t^n h^m g^u$$

Now, on comparing of coefficients of $t^n h^m g^u$ of the above equation, we get

$$[n+1]_{p,q}H_{n+1,m,u}^{k}(x,y,z;A;p,q)=\sqrt{2A}xH_{n,m,u}^{k}(px,qy,qz;A;p,q)$$

$$-(y+z)\sum_{r=0}^{k-1}\sum_{v=0}^{r}\frac{p^{\binom{k-1-r}{2}+\binom{r-v}{2}}q^{\binom{r}{2}+\binom{v}{2}}[k]_{p,q}!}{[k-r-1]_{p,q}![r-v]_{p,q}![v]_{p,q}!}H_{n+r+1-k,m+v-r,u-v}^{k}(px,py,pz;A;p,q).$$

Which the required relation (3.4).

Similarly way differentiating (2.2) with respect to h and g, we find

$$\sum_{n,m=0}^{\infty}[m+1]_{p,q}H_{n,m+1,u}^{k}(x,y,z;A;p,q)\, t^n h^m g^u$$

$$=\sqrt{2A}x\sum_{n,m,u=0}^{\infty}H_{n,m,u}^{k}(px,qy,qz;A;p,q)\, t^n h^m g^u$$

$$-[k]_{p,q}(y+z)\sum_{r=0}^{k-1}\sum_{v=0}^{r}\frac{p^{\binom{k-1-r}{2}+\binom{r-v}{2}}q^{\binom{r}{2}+\binom{v}{2}}[k-1]_{p,q}!}{[k-r-1]_{p,q}!\,[r-v]_{p,q}!\,[v]_{p,q}!}$$

$$\times\sum_{n,m,u=0}^{\infty}H_{n+r+1-k,m+v-r,u-v}^{k}(px,py,pz;A;p,q)\, t^n h^m g^u,$$

and

$$\sum_{n,m=0}^{\infty}[u+1]_{p,q}H_{n,m,u+1}^{k}(x,y,z;A;p,q)\, t^n h^m g^u$$

$$=\sqrt{2A}x\sum_{n,m,u=0}^{\infty}H_{n,m,u}^{k}(px,qy,qz;A;p,q)t^n h^m g^u$$

$$-[k]_{p,q}(y+z)\sum_{r=0}^{k-1}\sum_{v=0}^{r}\frac{p^{\binom{k-1-r}{2}+\binom{r-v}{2}}q^{\binom{r}{2}+\binom{v}{2}}[k-1]_{p,q}!}{[k-r-1]_{p,q}!\,[r-v]_{p,q}!\,[v]_{p,q}!}$$

$$\times\sum_{n,m,u=0}^{\infty}H_{n+r+1-k,m+v-r,u-v}^{k}(px,py,pz;A;p,q)\, t^n h^m g^u.$$

Now, on comparing of coefficients of $t^n h^m g^u$ of the above equations, we get

$$[m+1]_{p,q}H_{n,m+1,u}^{k}(x,y,z;A;p,q)=\sqrt{2A}xH_{n,m,u}^{k}(px,qy,qz;A;p,q)$$

$$-(y+z)\sum_{r=0}^{k-1}\sum_{v=0}^{r}\frac{p^{\binom{k-1-r}{2}+\binom{r-v}{2}}q^{\binom{r}{2}+\binom{v}{2}}[k]_{p,q}!}{[k-r-1]_{p,q}!\,[r-v]_{p,q}!\,[v]_{p,q}!}H_{n+r+1-k,m+v-r,u-v}^{k}(px,py,pz;A;p,q),$$

and

$$[u+1]_{p,q}H_{n,m,u+1}^{k}(x,y,z;A;p,q)=\sqrt{2A}xH_{n,m,u}^{k}(px,qy,qz;A;p,q)$$

$$-(y+z)\sum_{r=0}^{k-1}\sum_{v=0}^{r}\frac{p^{\binom{k-1-r}{2}+\binom{r-v}{2}}q^{\binom{r}{2}+\binom{v}{2}}[k]_{p,q}!}{[k-r-1]_{p,q}!\,[r-v]_{p,q}!\,[v]_{p,q}!}H_{n+r+1-k,m+v-r,u-v}^{k}(px,py,pz;A;p,q).$$

Which the required relations (3.5) and (3.6) respectively.

***Theorem* 3.3.** The (p,q)-Hermite matrix polynomials of three-index and three-variable $H_{n,m,u}^{k}(x,y,z;A;p,q)$ satisfies the following relation:

$$\frac{p^{\binom{n+m+u}{2}+\binom{n}{2}+\binom{m}{2}}q^{\binom{m+u}{2}+\binom{u}{2}}\left(x\sqrt{2A}\right)^{n+m+u}}{[n]_{p,q}!\,[m]_{p,q}!\,[u]_{p,q}!}$$

$$=\sum_{r=0}^{\left[\frac{n}{k}\right]}\sum_{s=0}^{\left[\frac{m}{k}\right]}\sum_{v=0}^{\left[\frac{u}{k}\right]}\frac{p^{\binom{r+s+v}{2}+\binom{kr}{2}+\binom{ks}{2}}q^{\binom{ks}{2}+\binom{kv}{2}}(y+z)^{r+s+v}[kr+ks+kv]_{p,q}!}{[r+s+v]_{p,q}![kr]_{p,q}![ks]_{p,q}![kv]_{p,q}!}H_{n-kr,m-ks,u-kv}^{k}(x,y,z;A;p,q)$$

. (3.7)

Proof. Using generating function of polynomials $H_{n,m,u}^{k}(x,y,z;A;p,q)$ and definition expression for function $exp_{p,q}\left(x\sqrt{2A}(g+h+t)\right)$ and $exp_{p,q}\left((y+z)(g+h+t)^k\right)$ we have

$$exp_{p,q}\left(x\sqrt{2A}(g+h+t)\right)=exp_{p,q}\left((y+z)(g+h+t)^k\right)\sum_{n,m,u=0}^{\infty}H_{n,m,u}^{k}(x,y,z;A;p,q)\, t^n h^m g^u$$

Using relations (1.6) and (1.14), we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{u=0}^{m}\frac{p^{\binom{n}{2}+\binom{n-m}{2}+\binom{m-u}{2}}q^{\binom{m}{2}+\binom{u}{2}}\left(x\sqrt{2A}\right)^{n}}{[n-m]_{p,q}!\,[m-u]_{p,q}!\,[u]_{p,q}!}t^{n-m}h^{m-u}g^{u}$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{v=0}^{s}\frac{p^{\binom{r}{2}+\binom{kr-ks}{2}+\binom{ks-kv}{2}}q^{\binom{ks}{2}+\binom{kv}{2}}(y+z)^{r}[kr]_{p,q}!}{[r]_{p,q}!\,[kr-ks]_{p,q}!\,[ks-kv]_{p,q}!\,[kv]_{p,q}!}t^{kr-ks}h^{ks-kv}g^{kv}$$

$$\times\sum_{n,m,u=0}^{\infty}H_{n,m,u}^{k}(x,y,z;A;p,q)\, t^n h^m g^u$$

On using relations (1.31) and (1.27), we get

$$\sum_{n,m,u=0}^{\infty}\frac{p^{\binom{n+m+u}{2}+\binom{n}{2}+\binom{m}{2}}q^{\binom{m+u}{2}+\binom{u}{2}}\left(x\sqrt{2A}\right)^{n+m+u}}{[n]_{p,q}!\,[m]_{p,q}!\,[u]_{p,q}!}t^n h^m g^u$$

$$=\sum_{n,m,u=0}^{\infty}\sum_{r,s,v=0}^{\infty}\frac{p^{\binom{r+s+v}{2}+\binom{kr}{2}+\binom{ks}{2}}q^{\binom{ks}{2}+\binom{kv}{2}}(y+z)^{r+s+v}[kr+ks+kv]_{p,q}!}{[r+s+v]_{p,q}!\,[kr]_{p,q}!\,[ks]_{p,q}!\,[kv]_{p,q}!}$$

$$\times H_{n,m,u}^{k}(x,y,z;A;p,q)t^{n+kr}h^{m+ks}g^{u+kv}$$

Comparing of the coefficients of $t^n h^m g^u$ of the above equation, we obtain the required relation (3.7).

Conclusion

The (p,q)-Hermite matrix polynomial of three variables is introduced and some of its recurrence relations are established in this paper. The approach used to introduce (p,q)-Hermite matrix polynomial depends on the scalar form of that function and the recurrence relations obtained are corresponding to the properties of the scalar function.

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مقالة بحثية

حول متعددات حدود مصفوفة (p, q)-هيرميت لثلاثة متغيرات

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المُلخّص

الهدف الرئيسي من هذه الورقة هو دراسة متعددات حدود مصفوفة (p, q)-هيرميت لثلاثة متغيرات باستخدام طريقة الدالة المولدة, حيث توضح هذه الدراسة فئة من متعددات حدود مصفوفة (p, q)-هيرميت بمساعدة وظائف التوليد مثل التمثيل الصريح واشتقاق بعض العلاقات التكرارية لها. يعطينا بناء هذه النتائج واشتقاقها فكرة عن كيفية التعامل مع الحسابات المعقدة التي تتضمن البارامترات q و p.

الكلمات المفتاحية: متعددات حدود مصفوفة (p, q)-هيرميت لثلاثة متغيرات؛ الدوال المولدة؛ حساب (p, q)؛ كثيرات الحدود المتعامدة؛ العلاقات التكرارية.

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