Electronic Journal of University of Aden for Basic and Applied Sciences

EJUA-BA Vol. 7 No. 2 (2026)

https://doi.org/10.47372/ejua-ba.2026.2.545

ISSN: 2708-0684

RESEARCH ARTICLE

LOGARITHMIC COEFFICIENTS AND HANKEL DETERMINANT FOR A NEW SUBCLASS OF CLOSE-TO-STAR FUNCTIONS ASSOCIATED WITH THE SINE FUNCTION

Safa'a Hamood Mohammed Al-Saqqaf[1,*]

[1] *Dept. of Mathematics, College of Humanities and Applied Sciences-Saber, University of Lahej, Yemen*

***Corresponding author: Safa'a Hamood Mohammed Al-Saqqaf; E-mail: safaalsaqqaf1@gmail.com**

Received: 01 June 2026 / Accepted: 16 June 2026 / Published online: 30 June 2026

Abstract

In this paper, we investigate the logarithmic coefficients for a new subclass of close-to-star functions associated with the sine function. We derive explicit formulas for the first six logarithmic coefficients γ_1 through γ_6 for functions in this class. We obtain upper bounds for the Hankel coefficients, Hankel determinants $H_{2,1}(f)$, $H_{2,2}(f)$, $H_{3,1}(f)$ and $H_{4,1}(f)$ associated with the class $\mathcal{CST}_0(\sin z)$. In addition, we derive sharp estimates for the Hankel determinant for the logarithmic coefficients $H_{2,1}(F_f/2)$ and $H_{2,2}(F_f/2)$ within the same class.

Keywords: Analytic functions; Close-to-star; Taylor coefficients; Logarithmic coefficients; Subordination; Hankel determinant.

1. Introduction

This study lies within the broader framework of geometric function theory, particularly the study of analytic and univalent functions in the unit disk.

Specifically, it focuses on the subclass of close-to-star functions, which are generalizations of starlike functions and are fundamental in understanding the geometric properties of analytic functions.

Let $\mathcal{A}$ denote the class of functions f that are analytic in the unit disk

$D = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, and that have a Maclaurin series expansion of the form

$$\mathrm{f}(\mathrm{z}) = \mathrm{z} + \sum_{m=2}^{\infty} a_m z^m \quad , \quad (\mathrm{z} \in D). \qquad (1.1)$$

A subclass of $\mathcal{A}$, denoted by $\mathcal{S}$, consists of functions that are univalent and normalized such that $f(0) = 0, f'(0) = 1$. Let $\mathcal{S}^*$ denote the subclass of $\mathcal{S}$ consisting of starlike functions, i.e., $f \in \mathcal{S}^*$ if and only if:

$$\mathcal{R}e\,\frac{zf'(z)}{f(z)} > 0 \;, \qquad z \in D. \qquad (1.2)$$

Let $\mathcal{B}$ denote the family of Schwarz functions $w(z)$, that are analytic in D given by

$$w(z) = \sum_{n=1}^{\infty} b_n z^n \quad , \quad (\mathrm{z} \in D)\,,$$

and satisfying $w(0) = 0$ and $|w(z)| < 1$ for all $z \in D$. Given analytic functions f and g in D, we say that f is subordinated to g, written $f \prec g$, if there exists a Schwarz function $w(z)$ such that $f(z) = g\big(w(z)\big)$, $z \in D$.

When g is univalent and $f(0) = g(0)$, then $f(D) \subset g(D)$.

For fixed constants A and B satisfying $-1 \leq B < A \leq 1$, denoted by $P\,[A, B]$, the family of functions

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n\,.$$

A function $p(z)$, analytic in the unit disk D, belongs to the Janowski class $P\,[A, B]$ if and only if

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}\,, \quad (\mathrm{z} \in D),$$

where $w(z)$ is Schwarz function. This class $P\,[A, B]$ is known as the class of Janowski and was introduced by [1].

Class $P\,[A, B] \subset P\,[1, -1] = P$, then it reduces to the class P(Carathéodory Class), the well-known class of functions with positive real part consists of functions p that satisfy $\mathcal{Re}\, p(z) > 0$ and $p\,(0) = 1$.

A function $f \in \mathcal{A}$ is called close-to-star if there exists $g \in \mathcal{S}^*$ and $\beta \in \mathbb{R}$ such

that

$$\mathcal{Re}\, \frac{e^{i\beta} f(z)}{g(z)} > 0 \;, z \in D. \tag{1.3}$$

Denote by $\mathcal{CST}$ the class of all close-to-star functions introduced by [2].

Note that $f \in \mathcal{CST}$ if and only if a function

$$F(z) = \int_0^z \frac{f(t)}{t}\, dt \;, z \in D,$$

is close-to-convex.

The class of close-to-star functions and its subclasses were intensively studied by various authors (e.g., MacGregor, Sakaguchi, Causey and Merke); for further references, see the reference by [3]. Given $g \in \mathcal{S}^*$ and $\beta \in \mathbb{R}$, let $\mathcal{CST}_\beta(g)$be the subclass of $\mathcal{CST}$ of all f satisfying (1.1).

In [4] defined and considered some familiar subsets of analytic functions associated with the sine function in the region of unit disk on the complex plane.

For example, the class $\mathcal{S}^*{}_{\sin}$ of starlike functions, $\mathcal{C}_{\sin}$ of convex functions and $\mathcal{R}_{\sin}$ of bounded turning functions:

$$\mathcal{S}^*{}_{\sin} = \left\{ \mathrm{f} \in \mathcal{A}\colon \frac{zf'(z)}{f(z)} \prec 1 + \sin z, \qquad z \in D \right\},$$

$$\mathcal{C}_{\sin} = \left\{ \mathrm{f} \in \mathcal{A}\colon 1 + \frac{zf''(z)}{f'(z)} \prec 1 + \sin z, \qquad z \in D \right\},$$

$$\mathcal{R}_{\sin} = \{ \mathrm{f} \in \mathcal{A}\colon f'(z) \prec 1 + \sin z, \qquad z \in D \}.$$

Also, they find the Hankel determinants of order three for these classes.

The class $\mathcal{S}^*{}_{\sin}$ was established by [5]and studied the radii problems.

In [6] defined and considered some classes with bounded turning function connect to the sine function, they studied upper bounds for the third and fourth Hankel determinant related to such classes.

In this paper, we consider a subclass of close-to-star functions associated with the sine function. The sine function, being a classical transcendental function, possesses rich analytic and geometric properties that motivate the present investigation.

We now introduce a subclass of close-to-star functions associated with the sine function follows:

Definition 1.1.

Let $D = \{z\colon z \in \mathbb{C}$ and $|z| < 1\}$ denote the open unit disk and let $k \in \mathcal{S}^*$be the Koebe function defined by

$$k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} nz^n = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}$$

Also, let $\varphi(z) = 1 + \sin z$, where φ is analytic and univalent in the open unit disk D and satisfies $\varphi(0) = 1$ and $\mathcal{Re}\, \varphi(z) > 0$.

We define the class $\mathcal{CST}(\sin z)$ as the set of all functions f in satisfying the subordination condition:

$$\frac{f(z)}{k(z)} \prec \varphi(z), \qquad z \in D. \tag{1.4}$$

This definition introduces a new subclass of close-to-star functions associated with the sine function, where the geometric behavior of the class is governed by the mapping properties of the dominant function $1 + \sin z$. However, most of these works have focused on coefficient estimates, geometric properties, or second Hankel determinants.

In contrast, this paper introduces distinct subclass $\mathcal{CST}_0(\sin z)$ defined by subordination associated with the sine function $\varphi(z) = 1 + \sin z$, and presents a comprehensive study of the logarithmic coefficients γ_1 to γ_6 as well as higher order Hankel determinants $H_{2,1}(F_f/2)$, $H_{2,2}(F_f/2)$, $H_{2,1}(f), H_{2,2}(f), H_{3,1}(f)$ and $H_{4,1}(f)$ while the general framework aligns with prior literature in geometric function theory, this paper distinguishes itself by providing new sharp bounds for both Taylor coefficients up to and logarithmic coefficients, which have received limited attention in previous research. This extension bridges a gap in the literature and contributes to a deeper understanding of analytic behavior in sine-associated subclass $\mathcal{CST}_0(\sin z)$.

The Fekete-Szegö inequality is a well-known result concerning the coefficients of univalent analytic functions, originally formulated by Fekete and Szegö in 1933 in connection with the Bieberbach conjecture. A related and important problem in the theory of univalent functions is the study of Hankel determinants, which have proven useful in the investigations of the singularities and power series with integral coefficients.

For the functions $f \in \mathcal{A}$ of the form (1.1), in [7] stated the ℓ^{th} Hankel determinant as

$$H_{\ell,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+\ell-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+\ell-2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+\ell-1} & a_{n+\ell} & \cdots & a_{n+2(\ell-1)} \end{vmatrix}, \tag{1.5}$$

$(a_1 = 1\; \ell, n \in N = \{1, 2, \cdot\cdot\cdot.\})$.

In particular, we have

$$H_{2,1}(f)=\begin{vmatrix}a_1 & a_2\\ a_2 & a_3\end{vmatrix}=a_3-{a_2}^2\ (a_1=1,n=1,\ell=2), \tag{1.6}$$

$$H_{2,2}(f)=\begin{vmatrix}a_2 & a_3\\ a_3 & a_4\end{vmatrix}=a_2a_4-{a_3}^2\ (\,n=2,\ell=2), \tag{1.7}$$

$$H_{3,1}(f)=\begin{vmatrix}a_1 & a_2 & a_3\\ a_2 & a_3 & a_4\\ a_3 & a_4 & a_5\end{vmatrix}=a_3a_5-a_3^3-a_4^2-a_2^2a_5+2a_2a_3a_4=a_3H_{2,2}(f)+a_4I+a_5H_{2,1}(f), \tag{1.8}$$

where $I=a_2a_3-a_4$,

and

$$H_{4,1}(f)=\begin{vmatrix}a_1 & a_2 & a_3 & a_4\\ a_2 & a_3 & a_4 & a_5\\ a_3 & a_4 & a_5 & a_6\\ a_4 & a_5 & a_6 & a_7\end{vmatrix}=-a_4\begin{vmatrix}a_2 & a_3 & a_4\\ a_3 & a_4 & a_5\\ a_4 & a_5 & a_6\end{vmatrix}+a_5\begin{vmatrix}a_1 & a_3 & a_4\\ a_2 & a_4 & a_5\\ a_3 & a_5 & a_6\end{vmatrix}-a_6\begin{vmatrix}a_1 & a_2 & a_4\\ a_2 & a_3 & a_5\\ a_3 & a_4 & a_6\end{vmatrix}+a_7\begin{vmatrix}a_1 & a_2 & a_3\\ a_2 & a_3 & a_4\\ a_3 & a_4 & a_5\end{vmatrix}$$

$$=-a_4[a_4(a_3a_5-a_4^2)-a_5(a_2a_5-a_3a_4)+a_6(a_2a_4-{a_3}^2\,)]$$

$$+a_5[a_3(a_3a_5-a_4^2)-a_5(a_5-a_2a_4)+a_6(a_4-a_2a_3)]$$

$$-a_6[a_3(a_2a_5-a_3a_4)-a_4(a_5-a_2a_4)+a_6(a_3-{a_2}^2\,)]+a_7H_{3,1}(f)$$

$$=a_7H_{3,1}(f)-a_6\rho_1+a_5\rho_2-a_4\rho_3, \tag{1.9}$$

where $\rho_1=a_3(a_2a_5-a_3a_4)-a_4(a_5-a_2a_4)+a_6(a_3-{a_2}^2\,)$,

$$\rho_2=a_3(a_3a_5-a_4^2)-a_5(a_5-a_2a_4)+a_6(a_4-a_2a_3),$$

$$\rho_3=a_4(a_3a_5-a_4^2)-a_5(a_2a_5-a_3a_4)+a_6(a_2a_4-{a_3}^2\,).$$

We note that $H_{2,1}(f)$ is the well-known Fekete-Szeg¨o functional [8], that is generalized as

$$\nabla(\mu,f\,)=|a_3-\mu{a_2}^2\,|, \tag{1.10}$$

for $\mu\in\mathbb{C}$.

Recently, the Hankel determinants of a function $f\in\mathcal{A}$ whose elements are logarithmic coefficients of $f\in\mathcal{A}$ have been introduced by [8, 9]

$$H_{\ell,n}(F_f/2)=\begin{vmatrix}\gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+\ell-1}\\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+\ell-2}\\ \vdots & \vdots & \cdots & \vdots\\ \gamma_{n+\ell-1} & \gamma_{n+\ell} & \cdots & \gamma_{n+2(\ell-1)}\end{vmatrix}.$$

The logarithmic coefficients are defined in the series form

$$\log\frac{f(z)}{z}=2\sum_{n=1}^{\infty}\gamma_nz^n\ . \tag{1.11}$$

Taking e of the exponential of both sides of (1.10), and differentiating logarithmically, we obtain

$$\gamma_1=\frac{1}{2}a_2\ \ , \tag{1.12}$$

$$\gamma_2=\frac{1}{2}\left(a_3-\frac{1}{2}a_2^2\right), \tag{1.13}$$

$$\gamma_3=\frac{1}{2}\left(a_4-a_2a_3+\frac{1}{3}a_2^3\right), \tag{1.14}$$

$$\gamma_4=\frac{1}{2}\left(a_5-a_2a_4+a_2^2a_3-\frac{1}{2}a_3^2-\frac{1}{4}a_2^4\right), \tag{1.15}$$

$$\gamma_5=\frac{1}{2}\left(a_6-a_2a_5-a_3a_4+a_2^2a_4+a_2a_3^2-a_3a_2^3+\frac{1}{5}a_2^5\right), \tag{1.16}$$

$$\gamma_6=\frac{1}{2}\left(a_7-a_2a_6-a_3a_5+a_2^2a_5-\frac{3}{2}a_2^2a_3^2-a_4a_2^3-\frac{1}{2}a_4^2+2a_2a_3a_4+\frac{1}{3}\,a_3^3+a_3a_2^4-\frac{1}{6}a_2^6\right). \tag{1.17}$$

The logarithmic coefficients have great importance, for instance, these coefficients helped [10] to solve Brennan’s conjecture for conformal mapping and estimation of the logarithmic coefficients can be transferred to the Taylor coefficients of univalent functions via the Lebedev–Milin inequalities [3] for details).

Some recent works on this problem that relate to the theory of univalent functions have been studied in [11-14] but only a few papers have been published for the class of starlike functions with respect to other points. Motivated by these works, in this paper, we obtain the upper bounds of the Taylor coefficients $|a_n|,n=2,3,4,5,6,7$.

In recent years, many papers have been devoted to finding the upper bounds for the second-order Hankel determinant $H_{2,2}$, for various subclasses of analytic functions and the upper bounds for the third and fourth-order Hankel determinants by many researchers [15-21].

The present work investigates a subclass of close-to-star functions associated with the sine function function. The sine function, being a classical transcendental function with rich analytic and geometric properties.

The study expands theoretical knowledge by deriving sharp estimates or properties of higher-order logarithmic coefficients and connecting classical functions with complex analysis by associating the sine function with a subclass of starlike functions.

The paper opens up new directions for analyzing function classes that are both geometrically meaningful and analytically rich.

Despite the extensive literature on subclasses associated with special functions, comparatively little attention has been devoted to logarithmic coefficients and higher-order Hankel determinants for close-to-star functions associated with the sine function. Motivated by this gap, the present work establishes explicit coefficient estimates and derives upper bounds for several Hankel determinants within this newly defined subclass.

2. Preliminary results

In this section, we give some lemmas to prove our main results.

Lemma 2.1. [3] For a function $p \in P$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, z \in D$

the sharp inequality $|c_n| \leq 2$ holds for each $n \geq 1$ and

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}.$$

Equality holds for the function $p(z) = \frac{1+z}{1-z}$.

Lemma 2.2. [22] Let $p \in P$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, z \in D$

and $\mu \in \mathbb{C}$. Then

$|c_n - \mu c_k c_{n-k}| \leq 2\max\{1, |2\mu - 1|\}, 1 \leq k \leq n - 1.$

If $|2\mu - 1| \geq 1$, then the inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$

or its rotations.

If $|2\mu - 1| < 1$, then the inequality is sharp for the function $p(z) = \frac{1+z^n}{1-z^n}$

or its rotations.

Lemma 2.3. [4] Let $p \in P$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, z \in D$

and $\alpha, \beta, \delta \in \mathbb{R}$. Then

$$|\alpha c_1^3 - \beta c_1 + \gamma c_3| \leq 2|\alpha| + 2|\beta - 2\alpha| + 2|\alpha - \beta + \delta|.$$

Lemma 2.4. [23] If $p \in P$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, z \in D,$ then

$$|c_2 - \mu c_1^2| \leq \begin{cases} -4\mu + 2 \ \ if\ \mu \leq 0 \\ 2\ if\ 0 \leq \mu \leq 1 \\ 4\mu - 2\ \text{ if } \mu \geq 1 \end{cases}$$

When $\mu < 0$ or $\mu > 1$, the equality holds if and only if $p(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < \mu < 1$, then equality holds if and only if $p(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $\mu = 0$, the equality holds if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1)$$

or one of its rotations. If $\mu = 1$, the equality holds if and only if p is the reciprocal of one of the functions such that the equality holds in the case of $\mu = 0$.

Lemma 2.5. [23].

If $p \in \mathcal{P}$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, z \in U$, then

$$|\mu c_3 - c_1^3| \leq \begin{cases} 2|\mu - 4| \text{ if } \mu \leq \frac{4}{3} \\ 2\mu\sqrt{\frac{\mu}{\mu - 1}} \text{ if } \frac{4}{3} < \mu \end{cases}$$

3. Taylor coefficients and Fekete- Szegö inequality for $f \in \mathcal{S}_{SC}^{*}(\sin z)$

Theorem 3.1.

If f is of the form (1.1) belongs to $\mathcal{CST}_0(\sin z)$, then

$$|a_2| \leq 3, |a_3| \leq 6, |a_4| \leq 10, |a_5| \leq 17$$

$$|a_6| \leq \frac{59}{2}, |a_7| \leq \frac{313}{6},$$

and

$$|a_3 - \mu a_2^2| \leq |2(1 - 2\mu) + 3 - 4\mu| + \frac{1}{2}\begin{cases} -4v + 2\ \ if\ v \leq 0 \\ 2\ if\ 0 \leq v \leq 1 \\ 4v - 2\ \text{ if } v \geq 1 \end{cases},$$

where $v = \frac{1}{2}(1 + \mu)$.

Proof. Since $f \in \mathcal{CST}_0(\sin z)$, from definition of subordination, there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$, and from (1.4) we have

$$\frac{f(z)}{k(z)} = 1 + \sin w(z), z \in D. \tag{3.1}$$

Assuming that

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

$$\begin{aligned}1 + w(z) = p(z)\big(1 - w(z)\big) \Rightarrow w(z)\big(1 + p(z)\big) \\ = p(z) - 1.\end{aligned}$$

This leads to

$$\begin{aligned}w(z) &= \frac{p(z)-1}{p(z)+1} \\ &= \frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{1}{2}{c_1}^2\right)z^2 \\ &+ \frac{1}{2}\left(c_3 - c_1c_2 + \frac{1}{4}{c_1}^3\right)z^3 \\ &+ \frac{1}{2}\left(c_4 - c_1c_3 - \frac{1}{2}{c_2}^2 + \frac{3}{4}{c_1}^2c_2 - \frac{1}{8}{c_1}^4\right)z^4 \\ &+ \frac{1}{2}\Big(c_5 - c_1c_4 - c_2c_3 - \frac{1}{2}c_2{c_1}^3 \\ &+ \frac{3}{4}{c_2}^2c_1 + \frac{3}{4}c_3{c_1}^2 + \frac{1}{16}{c_1}^5\Big)z^5 \\ &+ \frac{1}{2}\Big(c_6 - c_1c_5 - c_2c_4 - \frac{1}{2}{c_3}^2 \\ &+ \frac{1}{4}{c_2}^3 - \frac{3}{4}{c_1}^2{c_2}^2 \\ &+ \frac{3}{2}c_1c_2\,c_3 + \frac{3}{4}c_4{c_1}^2 + \frac{5}{16}{c_1}^4c_2 - \frac{1}{32}\ {c_1}^6\Big)z^6 + \cdots.\end{aligned}$$

Hence, from the right-hand side of (3. 1), we obtain

$$\begin{aligned}1 + \sin\ w(z) &= 1 + w(z) - \frac{\big(w(z)\big)^3}{3!} \\ &+ \frac{\big(w(z)\big)^5}{5!} - \cdots \\ &= 1 + \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}{c_1}^2\right)z^2 \\ &+ \left(\frac{4{c_1}^3}{48} - \frac{c_1c_2}{2} + \frac{c_3}{2}\right)z^3 \\ &+ \Big(\frac{c_4}{2} + \frac{5{c_1}^2c_2}{16} - \frac{{c_2}^2}{4} - \frac{c_1c_3}{2} \\ &- \frac{{c_1}^4}{32}\Big)z^4 \\ &+ \Big(\frac{c_5 - c_1c_4 - c_2c_3}{2} - \frac{1}{8}c_2{c_1}^3 \\ &+ \frac{5{c_2}^2c_1 + 5c_3{c_1}^2}{16} + \frac{1}{3840}{c_1}^5\Big)z^5 \\ &+ \Big(\frac{c_6 - c_1c_5 - c_2c_4}{2} - \frac{1}{4}{c_3}^2 \\ &+ \frac{5}{8}c_1c_2\,c_3 + \frac{5}{48}{c_2}^3 \\ &+ \frac{1}{32}{c_1}^4c_2 - \frac{1}{8}c_2{c_1}^3 + \frac{5}{16}{c_1}^2c_4 \\ &- \frac{3}{16}{c_1}^2{c_2}^2 + \frac{1}{96}\ {c_1}^6\Big)z^6 \\ &+ \cdots. \end{aligned} \tag{3.2}$$

On the other hand, since f of the form (1.1), this gives

$$\begin{aligned}f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + a_5z^5 + a_6z^6 \\ + a_7z^7 + \cdots,\end{aligned}$$

and

$$\begin{aligned}k(z) &= z + \sum_{n=2}^{\infty} nz^n \\ &= z + 2z^2 + 3z^3 + 4z^4 + 5z^5 \\ &+ 6z^6 + 7z^7 + \cdots.\end{aligned}$$

Further, we have from (3. 1) that

$$f(z) = k(z)(1 + \sin w(z)),$$

$$\begin{aligned}z + a_2z^2 + a_3z^3 + a_4z^4 + a_5z^5 + a_6z^6 + a_7z^7 \\ + \cdots \\ = (z + 2z^2 + 3z^3 + 4z^4 + 5z^5 \\ + 6z^6 + 7z^7 + \cdots)\end{aligned}$$

$$\begin{aligned}\Big[1 &+ \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}{c_1}^2\right)z^2 \\ &+ \left(\frac{5{c_1}^3}{48} - \frac{c_1c_2}{2} + \frac{c_3}{2}\right)z^3 \\ &+ \frac{1}{2}\left(c_4 - c_1c_3 - \frac{1}{2}{c_2}^2 + \frac{5}{8}{c_1}^2c_2 - \frac{1}{16}{c_1}^4\right)z^4 \\ &+ \Big(\frac{c_5 - c_1c_4 - c_2c_3}{2} - \frac{1}{8}c_2{c_1}^3 \\ &+ \frac{5{c_2}^2c_1 + 5c_3{c_1}^2}{16} + \frac{1}{3840}{c_1}^5\Big)z^5 \\ &+ \Big(\frac{c_6 - c_1c_5 - c_2c_4}{2} - \frac{1}{4}{c_3}^2 \\ &+ \frac{5}{8}c_1c_2\,c_3 \\ &+ \frac{5}{48}{c_2}^3 + \frac{1}{32}{c_1}^4c_2 - \frac{1}{8}c_2{c_1}^3 + \frac{5}{16}{c_1}^2c_4 - \frac{3}{16}{c_1}^2{c_2}^2 \\ &+ \frac{1}{96}{c_1}^6\Big)z^6 + \cdots\Big].\end{aligned} \tag{3.3}$$

Expanding the series and comparing the coefficients of z^n, n = 1, 2, 3, 4, 5,6,7 on both sides of (3.3) yields

$$a_2 = 2 + \frac{1}{2}c_1, \tag{3.4}$$

$$a_3 = c_1 + \frac{c_2}{2} - \frac{{c_1}^2}{4} + 3, \tag{3.5}$$

$$\begin{aligned}a_4 &= \frac{5{c_1}^3}{48} - \frac{c_1c_2}{2} + \frac{c_3}{2} + c_2 - \frac{{c_1}^2}{2} + \frac{3}{2}c_1 + 4 \\ &= \frac{5}{48}\left({c_1}^3 - \frac{24c_1c_2}{5} + \frac{24c_3}{5}\right) \\ &+ \left(c_2 - \frac{{c_1}^2}{2}\right) + \frac{3}{2}c_1 + 4,\end{aligned} \tag{3.6}$$

$$\begin{aligned}a_5 = 5 &+ \frac{c_4}{2} + \frac{5{c_1}^2c_2}{16} - \frac{{c_2}^2}{4} - \frac{c_1c_3}{2} - \frac{{c_1}^4}{32} \\ &+ 2\left(\frac{5{c_1}^3}{48} - \frac{c_1c_2}{2} + \frac{c_3}{2}\right) \\ &+ 2\left(\frac{c_2}{2} - \frac{{c_1}^2}{4}\right)\end{aligned}$$

$$+2c_1 = \frac{1}{2}(c_4 - c_1c_3) - \frac{c_2}{4}\left(c_2 - \frac{5}{4}{c_1}^2\right) - \frac{{c_1}^4}{32}$$
$$+ \frac{5}{24}\left({c_1}^3 - \frac{24c_1c_2}{5} + \frac{24c_3}{5}\right)$$
$$+\frac{3}{2}\left(c_2 - \frac{{c_1}^2}{2}\right) + 2c_1 + 5, \tag{3.7}$$

$$a_6 = \frac{c_5 - c_1c_4 - c_2c_3}{2} - \frac{1}{8}c_2{c_1}^3 + \frac{5{c_2}^2c_1 + 5c_3{c_1}^2}{16}$$
$$+ \frac{1}{3840}{c_1}^5$$

$$+2\left(\frac{c_4}{2} + \frac{5{c_1}^2c_2}{16} - \frac{{c_2}^2}{4} - \frac{c_1c_3}{2} - \frac{{c_1}^4}{32}\right)$$
$$+ 3\left(\frac{5{c_1}^3}{48} - \frac{c_1c_2}{2} + \frac{c_3}{2}\right)$$
$$+ 2\left(c_2 - \frac{{c_1}^2}{2}\right) + \frac{5}{2}c_1 + 6$$

$$= \frac{1}{2}(c_5 - c_1c_4) + \left(\frac{5c_3{c_1}^2}{16} + \frac{1}{3840}{c_1}^5 - \frac{1}{8}c_2{c_1}^3\right)$$
$$+ \left(\frac{5{c_2}^2c_1}{16} - \frac{c_2c_3}{2}\right) + (c_4 - c_1c_3)$$
$$+ \frac{5}{48}\cdot$$

$$3\left({c_1}^3 - \frac{24c_1c_2}{5} + \frac{24c_3}{5}\right) - \frac{c_2}{2}\left(c_2 - \frac{5}{4}{c_1}^2\right)$$
$$+ 2\left(c_2 - \frac{{c_1}^2}{2}\right) + \frac{5}{2}c_1 - \frac{{c_1}^4}{16} + 6$$
$$= \frac{1}{2}(c_5 - c_1c_4)$$

$$+ \frac{1}{3840}{c_1}^2({c_1}^3 - 480c_1c_2 + 1200c_3)$$
$$- \frac{1}{2}c_2\left(c_3 - \frac{5}{8}c_1c_2\right) + (c_4 - c_1c_3)$$
$$+ \frac{5}{2}c_1 - \frac{{c_1}^4}{16}$$

$$+ \frac{5}{16}\left({c_1}^3 - \frac{24c_1c_2}{5} + \frac{24c_3}{5}\right) - \frac{c_2}{2}\left(c_2 - \frac{5}{4}{c_1}^2\right)$$
$$+ 2\left(c_2 - \frac{{c_1}^2}{2}\right) + 6. \tag{3.8}$$

and

$$a_7 = \frac{c_6 - c_1c_5 - c_2c_4}{2} - \frac{1}{4}{c_3}^2 + \frac{5}{8}c_1c_2\,c_3 + \frac{5}{48}{c_2}^3$$
$$+ \frac{1}{32}{c_1}^4c_2 - \frac{1}{8}c_3{c_1}^3 + \frac{5}{16}{c_1}^2c_4$$

$$- \frac{3}{16}{c_1}^2{c_2}^2 + \frac{1}{96}{c_1}^6$$
$$+ 3\left(\frac{c_4}{2} + \frac{5{c_1}^2c_2}{16} - \frac{{c_2}^2}{4} - \frac{c_1c_3}{2}\right.$$
$$\left.- \frac{{c_1}^4}{32}\right)$$

$$+2\left(\frac{c_5 - c_1c_4 - c_2c_3}{2} - \frac{1}{8}c_2{c_1}^3\right.$$
$$\left.+ \frac{5{c_2}^2c_1 + 5c_3{c_1}^2}{16} + \frac{1}{3840}{c_1}^5\right)$$
$$+ 5\left(\frac{1}{2}c_2 - \frac{1}{4}{c_1}^2\right)$$

$$+4\left(\frac{5{c_1}^3}{48} - \frac{c_1c_2}{2} + \frac{c_3}{2}\right) + 3c_1 + 7$$
$$= \frac{1}{2}(c_6 - c_1c_5)$$
$$+ \left(\frac{5}{8}c_1c_2\,c_3 - \frac{1}{4}{c_3}^2 - \frac{1}{8}c_3{c_1}^3\right)$$

$$+ \left(\frac{5}{16}{c_1}^2c_4 - \frac{3}{16}{c_1}^2{c_2}^2\right) + \left(\frac{1}{32}{c_1}^4c_2 + \frac{1}{96}{c_1}^6\right)$$
$$+ \left(\frac{5}{48}{c_2}^3 - \frac{1}{2}c_2c_4\right)$$

$$+2\left(\frac{c_5 - c_1c_4 - c_2c_3}{2} - \frac{1}{8}c_2{c_1}^3\right.$$
$$\left.+ \frac{5{c_2}^2c_1 + 5c_3{c_1}^2}{16} + \frac{1}{3840}{c_1}^5\right)$$
$$+ 3c_1 + 7$$

$$+3\left(\frac{c_4}{2} + \frac{5{c_1}^2c_2}{16} - \frac{{c_2}^2}{4} - \frac{c_1c_3}{2} - \frac{{c_1}^4}{32}\right)$$
$$+ 4\left(\frac{5{c_1}^3}{48} - \frac{c_1c_2}{2} + \frac{c_3}{2}\right)$$
$$+ 5\left(\frac{1}{2}c_2 - \frac{1}{4}{c_1}^2\right)$$

$$= \frac{1}{2}(c_6 - c_1c_5) - \frac{1}{2}c_2\left(c_4 - \frac{5}{24}{c_2}^2\right) + 5$$
$$\cdot \frac{1}{2}\left(c_2 - \frac{{c_1}^2}{2}\right) + 3c_1 + 7$$
$$+ \frac{1}{32}{c_1}^4\left(c_2 + \frac{1}{3}{c_1}^2\right)$$

$$+ \frac{5}{16}{c_1}^2\left(c_4 - \frac{3}{5}{c_2}^2\right) + \frac{5}{12}\left({c_1}^3 - \frac{24c_1c_2}{5} + \frac{24c_3}{5}\right)$$
$$- \frac{1}{8}c_3({c_1}^3 - 5c_1c_2 + 2c_3)$$

$$+2\left[\frac{1}{2}(c_5 - c_1c_4)\right.$$
$$+ \frac{1}{3840}{c_1}^2({c_1}^3 - 480c_1c_2$$
$$+ 1200c_3)$$
$$\left.- \frac{1}{2}c_2\left(c_3 - \frac{5}{8}c_1c_2\right)\right]$$
$$+ 3\left[\frac{1}{2}(c_4 - c_1c_3) - \frac{c_2}{4}\left(c_2 - \frac{5}{4}{c_1}^2\right)\right.$$
$$\left.- \frac{{c_1}^4}{32}\right], \tag{3.9}$$

Using the triangle inequality and Lemma (2.2) in (3. 4), we get

$$|a_2| \leq \ 2 + \frac{1}{2} \cdot 2 = 3\,.$$

Now, applying Lemmas (2. 1) and (2.4) in (3. 5) and Lemmas (2.1), (2.3) and (2.4) in

(3. 6), respectively, implies that

$$|a_3| = \ \frac{1}{2}\left|c_2 - \frac{{c_1}^2}{2}\right| + |c_1| + 3 \leq 1 + 2 + 3 = 6,$$

where $\mu = \frac{1}{2}$.

$$\begin{aligned}|a_4| \leq \frac{5}{48}\left[2|1| + 2\left|\frac{24}{5} - 2(1)\right| + 2\left|1 - \frac{24}{5} + \frac{24}{5}\right|\right] \\ + 2 + \frac{3}{2} \cdot 2 + 4 \\ = \frac{5}{48}\left(4 + \frac{28}{5}\right) + 9 = 10,\end{aligned}$$

where $\alpha = 1\,, \beta = \frac{24}{5}\,, \delta = \frac{24}{5}\,, \mu = \frac{1}{2}$.

Consequently, by applying Lemmas (2.1), (2.2), (2.3) and (2.4) as well as the triangle inequality in (3.7), we obtain

$$\begin{aligned}|a_5| \leq \ \frac{1}{2} \cdot 2 + \ \frac{1}{2} \cdot 3 + 5 + \frac{16}{32} + \frac{5}{24} \cdot \frac{48}{5} + \frac{3}{2} \cdot 2 + 2 \cdot 2 \\ = 1 + \frac{3}{2} + 5 + \frac{1}{2} + 2 + 3 + 4 = 17.\end{aligned}$$

Using the triangle inequality and Lemmas (2.1), (2.2), (2.3) and (2.4) in (3. 8), we get

$$\begin{aligned}|a_6| \leq 1 + \frac{1}{3840} \cdot 4[2|1| + 2|480 - 2(1)| \\ + 2|1 - 480 + 1200|] + 2 + 2 \\ + 2\left(\frac{5}{2} - 1\right) + 3 \cdot \frac{5}{48}\left(4 + \frac{28}{5}\right) + 2 \\ \cdot 2 + \frac{5}{2} \cdot 2 + 6 \\ = 1 + \frac{5}{2} + 4 + 3 + 3 + 4 + 5 + 1 \\ + 6 = \frac{59}{2}\,,\end{aligned}$$

where $\mu_{\,1} = \mu_{\,2} = 1\,, \mu_{\,3} = \frac{1}{2}, \mu_{\,4} = \frac{5}{8}, \alpha_1 = 1, \beta_1 = 480\,, \delta_1 = 1200\,, \alpha_2 = 1,$

$$\beta_2 = \frac{24}{5}\,, \delta_2 = \frac{24}{5}\,.$$

Using the triangle inequality and Lemmas (2.1), (2.2), (2.3) and (2.4) in (3. 9), we get

$$\begin{aligned}|a_7| \leq 1 + 3 + \frac{5}{2} + \frac{5}{3} + 2 + 2\left(1 + \frac{5}{2} + 2\right) \\ + 3\left(1 + \frac{3}{2} + \frac{1}{2}\right) + 4 + 5 + 6 + 7 \\ = \frac{61}{6} + 11 + 9 + 22 = \frac{61}{6} + 42 = \frac{313}{6},\end{aligned}$$

where

$$\begin{aligned}\mu_1 = 1\,, \alpha_1 = 1, \beta_1 = 5\,, \delta_1 = 2\,, \mu_2 = \frac{3}{5}, \mu_3 = -\frac{1}{3}\,, \mu_4 \\ = \frac{5}{24}\,, \mu_5 = 1, \alpha_2 = 1,\end{aligned}$$

$$\begin{aligned}\beta_2 = 480\,, \delta_2 = 1200\,, \mu_6 = \frac{5}{8}\,, \mu_7 = 1, \mu_8 = \frac{5}{4}, \alpha_3 \\ = 1, \beta_3 = \frac{24}{5}\,, \delta_3 = \frac{24}{5}\,,\end{aligned}$$

$$\mu_5 = \frac{1}{2}\,.$$

$$\begin{aligned}a_3 - \mu a_2^2 = c_1 + \frac{c_2}{2} - \frac{{c_1}^2}{4} + 3 - \mu\left(2 + \frac{1}{2}c_1\right)^2 \\ = (1 - 2\mu)c_1 + \frac{c_2}{2} + 3 - 4\mu\end{aligned}$$

$$\begin{aligned}-\frac{1}{4}(1 + \mu){c_1}^2 = (1 - 2\mu)c_1 + 3 - 4\mu \\ + \frac{1}{2}\left[c_2 - \frac{1}{2}(1 + \mu){c_1}^2\right]\end{aligned}$$

where $v = \frac{1}{2}(1 + \mu)$.

By Lemmas (2.1) and (2.4), we get

$$\begin{aligned}|a_3 \ - \ \mu a_2^2| \leq |(1 - 2\mu)c_1 + 3 - 4\mu| \\ + \frac{1}{2}\left|\left[c_2 - \frac{1}{2}(1 + \mu){c_1}^2\right]\right| \\ \leq |2(1 - 2\mu) + 3 - 4\mu| \\ + \frac{1}{2}\begin{cases} -4v + 2 \ \ if \ v \leq 0 \\ \ 2\,if \ 0 \leq v \leq 1 \\ \ 4v - 2 \ \ \text{if}\, v \geq 1 \end{cases}.\end{aligned}$$

This completes the proof of Theorem (3.1).

Remark on sharpness:

The estimate for $a_2$is sharp and is attained by the extremal Carathéodory function

$$p(z) = \frac{1 + z}{1 - z} = 1 + 2\sum_{n=1}^{\infty}\ z^n,$$

we obtain $c_1 = 2$. Substituting into (3.4), we get $a_2 = 3$.

Or the extremal function

$$f(z) = \frac{z(1 + \sin z)}{(1 - z)^2} = z + 3z^2 + 5z^3 + \frac{41}{6}z^4 + \cdots.$$

For the higher-order coefficientsa_3, a_4, a_5, a_6 and a_7, the obtained inequalities provide upper bounds.

The sharpness of these bounds remains open further investigation.

Corollary 3.1.

If f is of the form (1.1) belongs to $\mathcal{CST}_0(\sin z)$, then

$$|H_{2,1}(f)| \leq 4.$$

The inequality is sharp.

Proof. Putting $\mu = 1$ in theorem 3.1, we obtain

$$\left|H_{2,1}(f)\right| = |a_3 \; - \; a_2^2| \leq |5-8| + \frac{1}{2} \cdot 2 = 4\,,$$

where $v = \frac{1}{2}(1+1) = 1$.

The estimate for $a_3 \; - \; a_2^2$is sharp and is attained by the extremal Carathéodory function

$$p(z) = \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n\,,$$

we obtain $c_1 = c_2 = 2$. Substituting into (3.4) and (3.5), we get

$$H_{2,1}(f) = 5 - 9 = -4 \Rightarrow \left|H_{2,1}(f)\right| = 4.$$

4. Hankel determinant for the Logarithmic coefficients of $f \in \mathcal{CST}_0(\sin z)$

Theorem 4.1.

If f is of the form (1.1) belongs to $\mathcal{CST}_0(\sin z)$, then

$$|\gamma_1| \leq \frac{3}{2}\;,|\gamma_2| \leq 1\;,|\gamma_3| \leq \frac{5}{6}\,,$$

$$|\gamma_4| \leq \frac{19}{4}\,,|\gamma_5| \leq \frac{9743}{240}\,,|\gamma_6| \leq \frac{335}{2}\,.$$

Proof. Putting (3. 4)- (3. 8) in (1.12)- (1.16), we obtain

$$\gamma_1 = \frac{a_2}{2} = 1 + \frac{1}{4}c_1 \tag{4.1}$$

$$\gamma_2 = \frac{1}{2}\Big(a_3 - \frac{1}{2}{a_2}^2\Big)$$

$$= \frac{1}{2}\Bigg[c_1 + \frac{c_2}{2} - \frac{{c_1}^2}{4} + 3$$

$$-\frac{1}{2}\Bigg(4 + \frac{{c_1}^2}{4}$$

$$+ 2c_1\Bigg)\Bigg]$$

$$= \frac{1}{2}\left[\frac{1}{2}\left(c_2 - \frac{3}{4}c_1^2\right) + 1\right], \tag{4.2}$$

$$\gamma_3 = \frac{1}{2}\Big(a_4 - a_2a_3 + \frac{1}{3}{a_2}^3\Big)$$

$$= \frac{1}{2}\Bigg[\Bigg[\frac{5}{48}\Big({c_1}^3 - \frac{24c_1c_2}{5} + \frac{24c_3}{5}\Big)$$

$$+ \Bigg(c_2 - \frac{{c_1}^2}{2}\Bigg)$$

$$+\frac{3}{2}c_1 + 4\Bigg] - \Big(2 + \frac{1}{2}c_1\Big)\Bigg(c_1 + \frac{c_2}{2} - \frac{{c_1}^2}{4} + 3\Bigg)$$

$$+\frac{1}{3}\Big(8 + 6c_1 + \frac{3}{2}{c_1}^2 + \frac{1}{8}{c_1}^3\Big)\Bigg]$$

$$= \frac{1}{2}\Bigg(\frac{13{c_1}^3}{48} - \frac{3c_1c_2}{4} + \frac{c_3}{2} + \frac{2}{3}\Bigg)$$

$$= \frac{1}{2}\Bigg[\frac{13}{48}\Big({c_1}^3 - \frac{36}{13}c_1c_2 + \frac{24}{13}c_3\Big)$$

$$+\frac{2}{3}\Bigg], \tag{4.3}$$

$$\gamma_4 = \frac{1}{2}\Bigg[\frac{1}{2}(c_4 - c_1c_3) - \frac{c_2}{4}\Big(c_2 - \frac{5}{4}{c_1}^2\Big) + 5 - \frac{{c_1}^4}{32}$$

$$+\frac{5}{24}\Big({c_1}^3 - \frac{24c_1c_2}{5} + \frac{24c_3}{5}\Big)$$

$$+\frac{3}{2}\Bigg(c_2 - \frac{{c_1}^2}{2}\Bigg) \quad + 2c_1$$

$$-\Bigg(2$$

$$+\frac{1}{2}c_1\Bigg)\Bigg(\frac{5}{48}\Big({c_1}^3 - \frac{24c_1c_2}{5} + \frac{24c_3}{5}\Big)$$

$$+\Bigg(c_2 - \frac{{c_1}^2}{2}\Bigg)$$

$$+\frac{3}{2}c_1 + 4\Bigg) + \Bigg(4 + 2c_1 + \frac{{c_1}^2}{4}\Bigg)\Bigg(c_1 + \frac{c_2}{2} - \frac{{c_1}^2}{4} + 3\Bigg)$$

$$-\frac{1}{2}\Bigg({c_1}^2 + 6c_1 + \frac{c_2}{2} - \frac{{c_2}^2}{4}$$

$$+c_1c_2 - \frac{3{c_1}^2}{2} + \frac{1}{16}{c_1}^4 - \frac{1}{2}{c_1}^3 + 3c_2 - \frac{{c_1}^2c_2}{4} + 9\Bigg)$$

$$-\frac{1}{4}\Bigg(16 + 16c_1 + 6{c_1}^2 + {c_1}^3 + \frac{1}{16}{c_1}^4\Bigg)\Bigg]$$

$$= \frac{1}{2}\Bigg(\frac{17}{2} + \frac{13{c_1}^2c_2}{16} + \frac{c_4}{2} - \frac{3c_1c_3}{4}$$

$$-\frac{3{c_2}^2}{8} - \frac{37{c_1}^4}{192} - \frac{{c_1}^3}{4} - \frac{3{c_1}^2}{4}\Bigg)$$

$$= \frac{1}{2}\Bigg[-\frac{37c_1}{192}\Big({c_1}^3 - \frac{156c_1\,c_2}{37} + \frac{144c_3}{37}\Big)$$

$$+\frac{1}{2}\Bigg(c_4 - \frac{3{c_2}^2}{4}\Bigg) + \frac{1}{2} + \frac{{c_1}^3}{4}$$

$$+\frac{3{c_1}^2}{4}\Bigg]. \tag{4.4}$$

$$\gamma_5 = \frac{1}{2}\Bigg(a_6 - a_2a_5 - a_3a_4 + a_2^2a_4 + a_2a_3^2$$

$$-a_3a_2^3 + \frac{1}{5}a_2^5\Bigg) = \frac{1}{2}\Bigg[\Bigg(\frac{1}{2}(c_5 - c_1c_4)$$

$$+\frac{1}{3840}{c_1}^2({c_1}^3 - 480c_1c_2 + 1200c_3)$$

$$-\frac{1}{2}c_2\Big(c_3 - \frac{5}{8}c_1c_2\Big) + (c_4 - c_1c_3)$$

$$+\frac{5}{16}\Big({c_1}^3 - \frac{24c_1c_2}{5} + \frac{24c_3}{5}\Big)$$

$$-\frac{c_2}{2}\Big(c_2 - \frac{5}{4}{c_1}^2\Big) + 2\Bigg(c_2 - \frac{{c_1}^2}{2}\Bigg)$$

$$+\frac{5}{2}c_1 - \frac{{c_1}^4}{16} + 6\Bigg)$$

$$-\left(2+\frac{1}{2}c_1\right)\Big(\frac{1}{2}(c_4-c_1c_3)-\frac{c_2}{4}\left(c_2-\frac{5}{4}{c_1}^2\right)+5$$

$$-\frac{{c_1}^4}{32}+\frac{5}{24}\Big({c_1}^3-\frac{24c_1c_2}{5}+\frac{24c_3}{5}\Big)$$

$$+\frac{3}{2}\left(c_2-\frac{{c_1}^2}{2}\right)+2c_1\Big)$$

$$-\Big(c_1+\frac{c_2}{2}-\frac{{c_1}^2}{4}$$

$$+3\Big)\Big(\frac{5}{48}\Big({c_1}^3-\frac{24c_1c_2}{5}+\frac{24c_3}{5}\Big)$$

$$+\frac{3}{2}c_1$$

$$+\left(c_2-\frac{{c_1}^2}{2}\right)+4\Big)$$

$$-\Big(c_1+\frac{c_2}{2}-\frac{{c_1}^2}{4}$$

$$+3\Big)\left(8+6c_1+\frac{3}{2}{c_1}^2+\frac{1}{8}{c_1}^3\right)$$

$$+\left(2+\frac{1}{2}c_1\right)$$

$$\Big({c_1}^2+6c_1+\frac{c_2}{2}-\frac{{c_2}^2}{4}+c_1c_2-\frac{3{c_1}^2}{2}+\frac{1}{16}{c_1}^4-\frac{1}{2}{c_1}^3$$

$$+3c_2-\frac{{c_1}^2c_2}{4}+9\Big)$$

$$+\frac{1}{5}\left(20{c_1}^2+32+40c_1+5{c_1}^3+\frac{5}{8}{c_1}^4+\frac{1}{32}{c_1}^5\right)\ \Big]$$

$$+\left(2c_1+\frac{{c_1}^2}{4}+4\right)$$

$$\Big(\frac{5}{48}\Big({c_1}^3-\frac{24c_1c_2}{5}+\frac{24c_3}{5}\Big)+\frac{3}{2}c_1+4+\left(c_2-\frac{{c_1}^2}{2}\right)\Big)$$

$$=\frac{1}{2}\Big(\frac{1}{2}c_5-\frac{3}{4}c_1c_4-\frac{3}{4}c_2c_3+\frac{9}{16}c_3{c_1}^2-\frac{23}{48}c_2{c_1}^3$$

$$+\frac{13}{16}{c_2}^2c_1+\frac{35}{256}{c_1}^5+\frac{5}{4}c_2{c_1}^2$$

$$-c_1c_3-\frac{1}{3}{c_1}^4-\frac{35}{192}{c_1}^3+\frac{7}{8}c_1c_2-\frac{7}{8}c_3+\frac{43}{2}c_1+\frac{2}{5}\Big)$$

$$=\frac{1}{2}\Big[\frac{1}{2}\left(c_5-\frac{3}{2}c_1c_4\right)-\frac{23}{24}c_2\left({c_1}^3-\frac{39}{46}c_1c_2+\frac{18}{23}c_3\right)$$

$$+\frac{7}{8}c_1\left(c_2-\frac{5{c_1}^2}{24}\right)$$

$$-c_1\left(c_3-\frac{5}{4}c_1c_2\right)$$

$$-\frac{3}{4}c_3\left(c_2-\frac{3{c_1}^2}{4}\right)+\frac{35}{256}{c_1}^5-\frac{1}{3}{c_1}^4$$

$$-\frac{7}{8}c_3+\frac{43}{2}c_1+\frac{2}{5}\Big], \tag{4.5}$$

to determine the required bound, we first derive the following relations

$$a_2a_6=c_5-\frac{1}{2}c_1c_4-c_2c_3+\frac{5}{8}c_3{c_1}^2+\frac{3}{8}{c_2}^2c_1$$

$$-\frac{1}{16}c_2{c_1}^3-\frac{59}{1920}{c_1}^5+2c_4$$

$$+\frac{1}{2}{c_1}^2c_2$$

$$-{c_2}^2+\frac{1}{4}c_1c_3+\frac{1}{12}{c_1}^4+\frac{1}{3}{c_1}^3-2c_1c_2+3c_3+4c_2$$

$$-2{c_1}^2+8c_1+12+\frac{1}{4}c_1c_5$$

$$-\frac{1}{4}c_4{c_1}^2$$

$$-\frac{1}{4}c_1c_2c_3+\frac{5}{32}{c_1}^2{c_2}^2+\frac{5}{32}{c_1}^3c_3-\frac{1}{16}{c_1}^4c_2$$

$$+\frac{1}{7680}{c_1}^6-\frac{1}{2}{c_1}^2c_3+\frac{5}{4}{c_1}^2, \tag{4.6}$$

$$a_3a_5=11c_1+\frac{1}{2}c_1c_4+\frac{2}{3}c_2{c_1}^3-\frac{3}{4}{c_2}^2c_1-\frac{3}{4}{c_1}^2c_3$$

$$-\frac{1}{12}{c_1}^5+\frac{29}{96}{c_1}^4-\frac{19}{16}{c_1}^2c_2$$

$$-\frac{1}{2}c_1c_3-\frac{1}{2}c_1c_2-\frac{5}{8}{c_1}^3-\frac{3}{2}{c_1}^2+7c_2+\frac{1}{4}c_2c_4$$

$$+\frac{7}{32}{c_1}^2{c_2}^2-\frac{1}{8}{c_2}^3-\frac{1}{4}c_1c_2c_3$$

$$-\frac{1}{2}c_1c_2-\frac{5}{8}{c_1}^3-\frac{3}{2}{c_1}^2+7c_2+\frac{1}{4}c_2c_4+\frac{7}{32}{c_1}^2{c_2}^2$$

$$-\frac{1}{8}{c_2}^3-\frac{1}{4}c_1c_2c_3-\frac{3}{32}{c_1}^4c_2$$

$$+\frac{1}{2}c_2c_3+\frac{3}{4}{c_2}^2-\frac{3}{8}c_4{c_1}^2+\frac{1}{8}{c_1}^3c_3+\frac{1}{128}{c_1}^6+\frac{3}{2}c_4$$

$$+15+3c_3\,, \tag{4.7}$$

$$a_2^2a_5=20+18c_1+\frac{9}{4}{c_1}^2-\frac{1}{6}{c_1}^3+\frac{5}{48}{c_1}^4-\frac{1}{96}{c_1}^5$$

$$+2c_4+c_1c_4+\frac{1}{8}c_4{c_1}^2-\frac{3}{8}{c_1}^2c_2$$

$$+\frac{3}{8}c_2{c_1}^3+\frac{5}{64}{c_1}^4c_2-{c_2}^2-\frac{1}{2}{c_2}^2c_1-\frac{1}{16}{c_1}^2{c_2}^2$$

$$-\frac{3}{4}c_3{c_1}^2-\frac{1}{8}{c_1}^3c_3-\frac{1}{128}{c_1}^6-c_1c_2$$

$$+4c_3+6c_2\ \ , \tag{4.8}$$

$$\frac{3}{2}a_2^2a_3^2=\frac{147}{8}{c_1}^2+\frac{3}{2}{c_2}^2+15c_1c_2-\frac{21}{6}{c_1}^4+72c_1$$

$$-\frac{9}{4}{c_1}^3+\frac{3}{4}{c_2}^2c_1+\frac{21}{8}{c_1}^2c_2+18c_2$$

$$-\frac{3}{8}c_2{c_1}^3+\frac{3}{32}{c_1}^2{c_2}^2+\frac{3}{128}{c_1}^6$$

$$-\frac{3}{32}{c_1}^4c_2+54, \tag{4.9}$$

$$\begin{aligned} a_4a_2^3 = \frac{7}{12}{c_1}^3 + 2c_1c_2 + 4c_3 + 8c_2 + 36c_1 + 11{c_1}^2 \\ + \frac{1}{16}{c_1}^4 + \frac{41}{384}{c_1}^5 - \frac{3}{2}{c_1}^2c_2 \\ + 3c_1c_3 - \frac{5}{8}c_2{c_1}^3 + \frac{3}{4}{c_1}^2c_3 \\ + \frac{1}{16}{c_1}^3c_3 - \frac{1}{16}{c_1}^4c_2 + 32\,, \end{aligned} \tag{4.10}$$

$$\begin{aligned} a_4^2 = \frac{25}{2304}{c_1}^6 + \frac{1}{4}{c_1}^2{c_2}^2 - \frac{5}{48}{c_1}^4c_2 + \frac{1}{4}{c_3}^2 \\ + \frac{5}{48}{c_1}^3c_3 - \frac{1}{2}c_1c_2c_3 + {c_2}^2 \\ - \frac{5}{2}{c_1}^2c_2 \\ - \frac{3}{2}{c_1}^2 + 12c_1 - c_1c_2 + 8c_2 - \frac{2}{3}{c_1}^3 - \frac{5}{48}{c_1}^5 + \frac{5}{16}{c_1}^4 \\ - {c_2}^2c_1 + \frac{1}{2}c_2{c_1}^3 + c_2c_3 \\ - \frac{1}{2}{c_1}^2c_3 + \frac{3}{2}c_1c_3 + c_3 + 16, \end{aligned} \tag{4.11}$$

$$\begin{aligned} a_2a_3a_4 = 2{c_1}^4 - \frac{15}{8}{c_1}^2c_2 + \frac{7}{4}c_1c_3 + 3c_1c_2 + \frac{3}{2}{c_1}^3 \\ + \frac{9}{4}{c_1}^2 + 23c_1 + \frac{3}{8}c_2{c_1}^3 \\ - \frac{1}{4}{c_2}^2c_1 + \frac{1}{2}c_2c_3 + {c_2}^2 + 10c_2 + 3c_3 + 24 \\ + \frac{7}{32}{c_1}^4c_2 - \frac{1}{8}{c_1}^2{c_2}^2 + \frac{1}{8}c_1c_2\,c_3 \\ - \frac{5}{64}{c_1}^6 - \frac{1}{16}{c_1}^3c_3 + \frac{1}{16}{c_1}^5\,, \end{aligned} \tag{4.12}$$

$$\begin{aligned} a_3^3 = 27c_1 + \frac{9}{4}{c_1}^2 - \frac{7}{2}{c_1}^3 - \frac{3}{2}{c_1}^2c_2 + \frac{3}{4}{c_2}^2c_1 + \frac{1}{8}{c_2}^3 \\ - \frac{3}{16}{c_1}^4 + \frac{3}{16}{c_1}^5 \\ + \frac{1}{64}{c_1}^6 + \frac{45}{2}c_2 + \frac{9}{4}{c_2}^2 + \frac{3}{32}{c_1}^4c_2 - \frac{3}{16}{c_1}^2{c_2}^2 \\ + 27\,, \end{aligned} \tag{4.13}$$

$$\begin{aligned} a_3a_2^4 = \frac{64}{3}c_1 + 8c_2 + 30{c_1}^2 + 48 + 8c_1c_2 + 5{c_1}^3 \\ + 3{c_1}^2c_2 - \frac{5}{16}{c_1}^4 + \frac{1}{2}c_2{c_1}^3 \\ - \frac{3}{16}{c_1}^5 + \frac{1}{32}{c_1}^4c_2 - \frac{1}{64}{c_1}^6\,, \end{aligned} \tag{4.14}$$

$$\begin{aligned} a_2^6 = 64 + 96c_1 + 60{c_1}^2 + 20{c_1}^3 + \frac{15}{4}{c_1}^4 + \frac{3}{8}{c_1}^5 \\ + \frac{1}{64}{c_1}^6, \end{aligned} \tag{4.15}$$

Substituting (3. 9) and (4.6)- (4.15) in (1.17), we obtain

$$\begin{aligned} \gamma_6 = \frac{1}{2}\Bigg(\frac{1}{2}c_6 - \frac{3}{4}c_1c_5 - \frac{3}{4}c_2c_4 + \frac{13}{8}c_1c_2\,c_3 + \frac{13}{48}{c_2}^3 \\ - \frac{3}{8}{c_3}^2 + \frac{187}{192}{c_1}^4c_2 \\ - \frac{37}{48}c_3{c_1}^3 + \frac{17}{16}{c_1}^2c_4 - \frac{37}{32}{c_1}^2{c_2}^2 - \frac{293}{1440}{c_1}^6 \\ + \frac{67}{48}c_2{c_1}^3 - \frac{5}{384}{c_1}^5 \\ + \frac{1}{8}{c_1}^2c_2 - \frac{5}{2}c_1\,c_3 - 73c_1 + \frac{119}{32}{c_1}^4 + \frac{145}{24}{c_1}^3 + c_1c_2 \\ - \frac{3}{4}{c_2}^2 + \frac{3}{2}c_3 + 3c_2 \\ - \frac{1}{8}{c_1}^2 + \frac{1}{3}\Bigg) = \frac{1}{2}\Bigg[\frac{1}{2}\left(c_6 - \frac{3}{2}c_1c_5\right) \\ - \frac{37}{48}c_3\left({c_1}^3 - \frac{78}{37}\,c_1c_2 + \frac{18}{37}c_3\right) \\ + \frac{17}{16}{c_1}^2\left(c_4 - \frac{37}{34}{c_2}^2\right) - 73c_1 \\ + \frac{187}{192}{c_1}^4\left(c_2 - \frac{586}{2805}{c_1}^2\right) \\ - \frac{3}{4}c_2\left(c_4 - \frac{13}{36}{c_2}^2\right) \\ +3\left(c_2 - \frac{1}{24}{c_1}^2\right) + \frac{145}{24}\left({c_1}^3 + \frac{24}{145}\,c_1c_2 + \frac{36}{145}c_3\right) \\ + \frac{67}{48}{c_1}^3\left(c_2 - \frac{5}{536}{c_1}^2\right) \\ + \frac{119}{32}c_1\left({c_1}^3 + \frac{4}{119}\,c_1c_2 - \frac{80}{119}c_3\right) \\ - \frac{3}{4}{c_2}^2 + \frac{1}{3}\Bigg]. \end{aligned} \tag{4.16}$$

The bounds of $|\gamma_1|$, $|\gamma_2|$, $|\gamma_3|$, $|\gamma_4|$, $|\gamma_5|$ and $|\gamma_6|$follow from Lemmas (2.1), (2.2), (2.3) and (2.4) On the other hand, rearranging the terms in (4.1), (4.2), (4.3), (4.4), (4.5) and (4.16), we get

$$|\gamma_1| \ = \left|1 + \frac{1}{4}c_1\ \right| \leq 1 + \frac{1}{2} = \frac{3}{2}.$$

$$|\gamma_2| = \frac{1}{2}\left|\frac{1}{2}\left(c_2 - \frac{3}{4}c_1^2\right) + 1\right| \leq \frac{1}{2}\left(\frac{1}{2}\cdot 2 + 1\right) = 1\,,$$

where $\mu \ = \frac{3}{4}$.

$$\begin{aligned} |\gamma_3| = \frac{1}{2}\left|\left[\frac{13}{48}\left({c_1}^3 - \frac{36}{13}c_1c_2 + \frac{24}{13}c_3\right) + \frac{2}{3}\right]\right| \\ \leq \ \frac{1}{2}\left[\frac{13}{48}\left(2|1| + 2\left|\frac{36}{13} - 2\right| + 2\left|1 - \frac{36}{13} + \frac{24}{13}\right|\right) + \frac{2}{3}\right] \\ = \ \frac{1}{2}\left[\frac{13}{48}\left(2 + \frac{22}{13}\right) + \frac{2}{3}\right] = \frac{5}{6}, \end{aligned}$$

where $\alpha \ = 1, \beta \ = \frac{36}{13}\ , \delta \ = \frac{24}{13}$.

$$\begin{aligned} |\gamma_4| = \frac{1}{2}\Bigg|- \frac{37c_1}{192}\left({c_1}^3 - \frac{156c_1\,c_2}{37} + \frac{144c_3}{37}\right) \\ + \frac{1}{2}\left(c_4 - \frac{3{c_2}^2}{4}\right) + \frac{1}{2} + \frac{{c_1}^3}{4} + \frac{3{c_1}^2}{4}\Bigg| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\left[\ \frac{37}{96}\left(2|1|+2\left|\frac{156}{37}-2\right|+2\left|1-\frac{156}{37}+\frac{144}{37}\right|\right)\right.\\
&\qquad \left.+\frac{1}{2}\cdot 2++\frac{1}{2}+2+3\right]\\
&\qquad =\frac{1}{2}\left(3+1+\frac{1}{2}+5\right)=\frac{19}{4},
\end{aligned}
$$

where $\mu=\frac{3}{4}, \alpha=1, \beta=\frac{156}{37}, \delta=\frac{144}{37}$.

$$
\begin{aligned}
|\gamma_5| = \frac{1}{2}\Bigg|&\frac{1}{2}\left(c_5-\frac{3}{2}c_1c_4\right)\\
&-\frac{23}{24}c_2\left(c_1{}^3-\frac{39}{46}\,c_1c_2+\frac{18}{23}c_3\right)\\
&+\frac{7}{8}c_1\left(c_2-\frac{5c_1{}^2}{24}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-c_1\left(c_3-\frac{5}{4}c_1c_2\right)-\frac{3}{4}c_3\left(c_2-\frac{3c_1{}^2}{4}\right)+\frac{35}{256}c_1{}^5\\
&\qquad -\frac{1}{3}c_1{}^4-\frac{7}{8}c_3+\frac{43}{2}c_1+\frac{2}{5}\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2}\left[\frac{1}{2}\cdot 2\,(3-1)+\frac{23}{12}\left(2+\frac{53}{23}+\frac{43}{23}\right)+\frac{7}{8}\cdot 4+2\right.\\
&\qquad \cdot 2\left(\frac{5}{2}-1\right)+\frac{3}{4}\cdot 2\cdot 2+\frac{35}{256}\cdot 32+\frac{1}{3}\\
&\qquad \left.\cdot 16+\frac{7}{8}\cdot 2+\frac{43}{2}\cdot 2+\frac{2}{5}\right]\\
&\qquad =\frac{1}{2}\left[2+\frac{71}{6}+\frac{7}{2}+6+3+\frac{35}{8}+\frac{16}{3}\right.\\
&\qquad \left.+\frac{7}{4}+43+\frac{2}{5}\right]
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{1}{2}\left(\frac{35}{8}+\frac{71}{6}+\frac{7}{2}+\frac{16}{3}+\frac{7}{4}+\frac{2}{5}+54\right)\\
&\qquad =\frac{1}{2}\left(\frac{3263}{120}+54\right)=\frac{1}{2}\cdot\frac{9743}{120}\\
&\qquad =\frac{9743}{240},
\end{aligned}
$$

where $\mu_1=\frac{3}{2}, \alpha=1, \beta=\frac{39}{46}, \delta=\frac{18}{23}, \mu_2=\frac{5}{24}, \mu_3=\frac{5}{4}, \mu_4=\frac{3}{4}$.

$$
\begin{aligned}
|\gamma_6| = \frac{1}{2}\Bigg|\Bigg[&\frac{1}{2}\left(c_6-\frac{3}{2}c_1c_5\right)\\
&-\frac{37}{48}c_3\left(c_1{}^3-\frac{78}{37}\,c_1c_2+\frac{18}{37}c_3\right)\\
&+\frac{17}{16}c_1{}^2\left(c_4-\frac{37}{34}c_2{}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{187}{192}c_1{}^4\left(c_2-\frac{586}{2805}c_1{}^2\right)-\frac{3}{4}c_2\left(c_4-\frac{13}{36}c_2{}^2\right)\\
&\qquad +\frac{145}{24}\left(c_1{}^3+\frac{24}{145}\,c_1c_2+\frac{36}{145}c_3\right)
\end{aligned}
$$

$$
\begin{aligned}
&+3\left(c_2-\frac{1}{24}c_1{}^2\right)+\frac{67}{48}c_1{}^3\left(c_2-\frac{5}{536}c_1{}^2\right)\\
&\qquad +\frac{119}{32}c_1\left(c_1{}^3+\frac{4}{119}\,c_1c_2-\frac{80}{119}c_3\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3}{4}c_2{}^2-73c_1+\frac{1}{3}\Bigg]\Bigg|\\
&\qquad \leq\frac{1}{2}\left[\frac{1}{12}\cdot 2\cdot 2+2\cdot\frac{37}{48}\left(2|1|\right.\right.\\
&\qquad +2\left|\frac{78}{37}-2(1)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\left.+2\left|1-\frac{78}{37}+\frac{18}{37}\right|\right)+\frac{17}{16}\cdot 4\cdot 2\left(\frac{37}{17}-1\right)+\frac{187}{192}\cdot 16\\
&\qquad \cdot 2+\frac{3}{4}\cdot 2\cdot 2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{145}{24}\left(2|1|+2\left|-\frac{24}{145}-2(1)\right|+2\left|1+\frac{24}{145}+\frac{36}{145}\right|\right)\\
&\qquad +3\cdot 2+\frac{67}{48}\cdot 8\cdot 2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{119}{32}\cdot 2\left(2|1|+2\left|-\frac{4}{119}-2(1)\right|\right.\\
&\qquad \left.+2\left|1+\frac{4}{119}-\frac{80}{119}\right|\right)+\frac{3}{4}\\
&\qquad \left.\cdot 4+73\cdot 2+\frac{1}{3}\right]
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{1}{2}\left[2+\frac{37}{24}\left(2+\frac{8}{37}+\frac{46}{37}\right)+10+\frac{187}{6}+3\right.\\
&\qquad +\frac{145}{24}\left(2+\frac{628}{145}+\frac{82}{29}\right)+6+\frac{67}{3}
\end{aligned}
$$

$$
\begin{aligned}
&\left.+\frac{101}{2}+\frac{448}{3}\right]=\frac{1}{2}\left[21+\frac{16+166+448+67}{3}+\frac{187}{6}\right.\\
&\qquad \left.+\frac{101}{2}\right]=\frac{335}{2},
\end{aligned}
$$

where $\mu_1=\frac{3}{2}, \alpha_1=1, \beta_1=\frac{78}{37}, \delta_1=\frac{18}{37}, \mu_2=\frac{37}{34}, \mu_3=\frac{586}{2805}, \mu_4=\frac{13}{36}, \alpha_2=1,$

$$
\begin{aligned}
\beta_2=-\frac{24}{145}, \delta_2=\frac{36}{145}, \mu_5=\frac{1}{24}, \mu_6=\frac{5}{536}, \alpha_3\\
=1, \beta_3=-\frac{4}{119}, \delta_3=-\frac{80}{119}.
\end{aligned}
$$

Remark on sharpness:

The estimate for γ_1 is sharp and is attained by the extremal Carathéodory function

$$
p(z)=\frac{1+z}{1-z}=1+2\sum_{n=1}^{\infty}\ z^n,
$$

we obtain $c_1=2$. Substituting into (4.1), we get $\gamma_1=\frac{3}{2} \Rightarrow |\gamma_1|=\frac{3}{2}$.

The estimate for γ_2 is sharp. Indeed, by choosing the extremal Carathéodory function

$$
p(z)=\frac{1+z^2}{1-z^2}=1+2z^2+\cdots,
$$

we obtain $c_1=0, c_2=2$. Substituting into (4.2), we get $\gamma_2=1 \Rightarrow |\gamma_2|=1$.

The estimate for γ_3 is sharp. Indeed, by choosing the extremal Carathéodory function

$$p(z)=\frac{1+z^3}{1-z^3}=1+2z^3+\cdots,$$

we obtain $c_1=c_2=0, c_3=2$. Substituting into(4.3), we get $\gamma_3=\frac{5}{6} \Rightarrow |\gamma_3|=\frac{5}{6}$.

The estimate for $\gamma_4$is sharp. Indeed, by choosing the extremal Carathéodory function

$$p(z)=\frac{1+z^4}{1-z^4}=1+2z^4+\cdots,$$

we obtain $c_1=c_2=c_3=0, c_4=2$. Substituting into(4.4), we get $\gamma_4=\frac{19}{4} \Rightarrow |\gamma_4|=\frac{19}{4}$.

However, the sharpness of the bounds for the higher-order logarithmic coefficients γ_5 and γ_6 remains open and requires further investigation.

Theorem 4.2.

If f is of the form (1.1) belongs to $\mathcal{CST}_0(\sin z)$, then

$$\left|H_{2,1}(F_f/2)\right| \le \frac{141+2\sqrt{141}}{94}.$$

Proof. In view of (1.12), (1.13), (1.14), we have

$$\begin{aligned}
H_{2,1}(F_f/2) &= \gamma_1\gamma_3-{\gamma_2}^2\\
&= \frac{1}{2}\left(1\right.\\
&\quad \left.+\frac{1}{4}c_1\right)\left(\frac{13}{48}\left({c_1}^3-\frac{36}{13}c_1c_2+\frac{24}{13}c_3\right)\right.\\
&\quad \left.+\frac{2}{3}\right)\\
&- \left(\frac{1}{2}\left[\frac{1}{2}\left(c_2-\frac{3}{4}c_1^2\right)+1\right]\right)^2\\
&= \frac{1}{2}\left(\frac{13}{48}\left({c_1}^3-\frac{36}{13}c_1c_2+\frac{24}{13}c_3\right)+\frac{2}{3}\right.\\
&\quad +\frac{13}{192}{c_1}^4+\frac{1}{6}c_1\\
&\quad \left.-\frac{3}{16}{c_1}^2c_2+\frac{1}{8}c_1c_3\right)\\
&\quad -\frac{1}{4}\left(\frac{1}{4}{c_2}^2+\frac{9}{64}{c_1}^4-\frac{3}{8}{c_1}^2c_2+1\right.\\
&\quad \left.+c_2-\frac{3}{4}{c_1}^2\right)\\
&= \frac{13}{96}\left({c_1}^3-\frac{36}{13}c_1c_2+\frac{24}{13}c_3\right)+\frac{1}{768}c_1(48c_3-{c_1}^3)\\
&\quad -\frac{1}{4}\left(c_2-\frac{3}{4}{c_1}^2\right)+\frac{1}{12}c_1\\
&\quad -\frac{1}{16}{c_2}^2+\frac{1}{12}. \qquad (4.17)
\end{aligned}$$

By applying the triangle inequality in (4.17) as well as Lemmas (2.1), (2.3), (2.4) and (2.5), we get the desired inequality.

$$\begin{aligned}
\left|\gamma_1\gamma_3-{\gamma_2}^2\right| &\le \frac{13}{96}\left(2|1|+2\left|\frac{36}{13}-2\right|\right.\\
&\quad \left.+2\left|1-\frac{36}{13}+\frac{24}{13}\right|\right)\\
&\quad +\frac{1}{384}\left(2\cdot 48\sqrt{\frac{48}{47}}\right)+\frac{1}{4}\cdot 2+\frac{1}{6}\\
&\quad +\frac{1}{16}\cdot 4+\frac{1}{12}\\
&= \frac{1}{2}+\sqrt{\frac{141}{47}}+\frac{1}{2}+\frac{1}{6}+\frac{1}{4}+\frac{1}{12}\\
&= \frac{141+2\sqrt{141}}{94},
\end{aligned}$$

where $\alpha=1, \beta=\frac{36}{13}, \delta=\frac{24}{13}, \mu_1=48, \mu_2=\frac{3}{4}$.

Theorem 4.3.

If f is of the form (1.1) belongs to $\mathcal{CST}_0(\sin z)$, then

$$\left|H_{2,2}(F_f/2)\right| \le \frac{87}{16}.$$

Proof. In view of (1.13), (1.14), (1.15), we have

$$\begin{aligned}
H_{2,2}(F_f/2) &= \gamma_2\gamma_4-{\gamma_3}^2\\
&= \frac{1}{4}\left(\frac{1}{2}\left(c_2-\frac{3}{4}c_1^2\right)+1\right)\left(\frac{1}{2}\right.\\
&\quad +\frac{13{c_1}^2c_2}{16}+\frac{c_4}{2}-\frac{3c_1c_3}{4}-\frac{3{c_2}^2}{8}\\
&\quad \left.-\frac{37{c_1}^4}{192}-\frac{{c_1}^3}{4}-\frac{3{c_1}^2}{4}\right)\\
&\quad -\left(\frac{13}{48}\left({c_1}^3-\frac{36}{13}c_1c_2+\frac{24}{13}c_3\right)+\frac{2}{3}\right)^2\\
&= \frac{1}{4}\left(\frac{1}{4}c_2+\frac{1}{4}c_2c_4-\frac{3}{16}{c_2}^3-\frac{3}{8}c_1c_2c_3+\frac{35}{64}{c_1}^2{c_2}^2\right.\\
&\quad -\frac{77}{192}{c_1}^4c_2-\frac{3}{16}{c_1}^2c_4+\frac{9}{32}{c_1}^3c_3\\
&\quad -\frac{1}{8}{c_1}^3c_2+\frac{7}{16}{c_1}^2c_2-\frac{15}{16}{c_1}^2+\frac{3}{32}{c_1}^5+\frac{37}{512}{c_1}^6+\frac{1}{2}\\
&\quad +\frac{1}{2}c_4-\frac{3}{8}{c_2}^2-\frac{3}{4}c_1c_3\\
&\quad \left.-\frac{1}{4}{c_1}^3+\frac{37}{192}{c_1}^4\right)\\
&\quad -\frac{1}{4}\left(\frac{169}{2304}{c_1}^6+\frac{9}{16}{c_1}^2{c_2}^2\right.\\
&\quad -\frac{13}{32}{c_1}^4c_2+\frac{1}{4}{c_3}^2+\frac{1}{4}{c_3}^2+\frac{2}{3}c_3\\
&\quad \left.+\frac{13}{48}{c_1}^3c_3-\frac{3}{4}c_1c_2c_3+\frac{13}{72}{c_1}^3-\frac{1}{2}c_1c_2+\frac{4}{9}\right)\\
&= \frac{1}{16}c_2+\frac{1}{16}c_2c_4-\frac{3}{64}{c_2}^3\\
&\quad -\frac{1}{256}{c_1}^2{c_2}^2
\end{aligned}$$

$$+\frac{1}{768}{c_1}^4c_2-\frac{3}{64}{c_1}^2c_4+\frac{1}{384}{c_1}^3c_3+\frac{3}{32}c_1c_2c_3$$
$$-\frac{1}{32}{c_1}^3c_2+\frac{7}{64}{c_1}^2c_2-\frac{15}{64}{c_1}^2$$

$$-\frac{5}{18432}{c_1}^6-\frac{1}{16}{c_3}^2+\frac{3}{128}{c_1}^5+\frac{1}{8}c_4-\frac{3}{32}{c_2}^2$$
$$-\frac{3}{16}c_1c_3-\frac{31}{288}{c_1}^3+\frac{1}{8}c_1c_2$$

$$+\frac{37}{768}{c_1}^4-\frac{1}{6}c_3+\frac{1}{72}$$
$$=\left[\left(\frac{1}{384}{c_1}^3c_3-\frac{1}{16}{c_3}^2+\frac{3}{32}c_1c_2c_3\right)\right.$$
$$+\left(\frac{1}{8}c_4-\frac{3}{32}{c_2}^2\right)$$

$$+\left(-\frac{1}{256}{c_1}^2{c_2}^2-\frac{3}{64}{c_1}^2c_4\right)$$
$$+\left(\frac{1}{8}c_1c_2-\frac{31}{288}{c_1}^3-\frac{1}{6}c_3\right)$$
$$+\left(\frac{1}{16}c_2-\frac{15}{64}{c_1}^2\right)$$

$$+\left(\frac{1}{768}{c_1}^4c_2-\frac{5}{18432}{c_1}^6\right)$$
$$+\left(\frac{37}{768}{c_1}^4+\frac{7}{64}{c_1}^2c_2-\frac{3}{16}c_1c_3\right)$$
$$-\frac{1}{16}{c_3}^2+\frac{1}{72}$$

$$\left.+\left(\frac{1}{16}c_2c_4-\frac{3}{64}{c_2}^3\right)+\frac{3}{128}{c_1}^5\right]$$

$$=\left[\frac{1}{384}c_3({c_1}^3+36c_1c_2-24c_3)+\frac{1}{8}\left(c_4-\frac{3}{4}{c_2}^2\right)\right.$$
$$-\frac{3}{64}{c_1}^2\left(c_4+\frac{1}{12}{c_2}^2\right)+\frac{3}{128}{c_1}^5$$

$$-\frac{31}{288}\left({c_1}^3-\frac{36}{31}c_1c_2+\frac{48}{31}c_3\right)+\frac{1}{16}\left(c_2-\frac{15}{4}{c_1}^2\right)$$
$$+\frac{1}{768}{c_1}^4\left(c_2-\frac{5}{24}{c_1}^2\right)$$

$$+\frac{37}{768}c_1\left({c_1}^3+\frac{84}{37}c_1c_2-\frac{144}{37}c_3\right)$$
$$+\frac{1}{16}c_2\left(c_4-\frac{3}{4}{c_2}^2\right)$$
$$\left.-\frac{1}{16}{c_3}^2+\frac{1}{72}\right],\qquad(4.18)$$

by applying the triangle inequality in (4.18) as well as Lemmas (2.1), (2.2), (2.3) and (2.4), we get the desired inequality.

$$|H_{2,2}(F_f/2)|\leq\frac{1}{384}\cdot2(2+78+26)+\frac{1}{8}\cdot2+\frac{3}{64}\cdot4$$
$$\cdot2\left(1+\frac{1}{6}\right)$$

$$+\frac{31}{288}\left(2+\frac{52}{31}+\frac{86}{31}\right)+\frac{1}{16}\cdot2\left(\frac{15}{2}-1\right)+\frac{1}{768}\cdot16\cdot2$$
$$+\frac{37}{768}\cdot2\left(2+\frac{316}{37}+\frac{46}{37}\right)$$

$$+\frac{1}{16}\cdot2\cdot2+\frac{3}{128}\cdot32+\frac{1}{16}\cdot8+\frac{1}{72}$$
$$=\frac{53}{96}+\frac{1}{4}+\frac{7}{16}+\frac{25}{36}+\frac{13}{16}+\frac{1}{24}$$
$$+\frac{109}{96}+\frac{1}{4}$$

$$+\frac{3}{4}+\frac{1}{2}+\frac{1}{72}=\frac{529}{144}+\frac{127}{72}=\frac{87}{16}.$$

$$=\left[\frac{1}{384}c_3({c_1}^3+36c_1c_2-24c_3)+\frac{1}{8}\left(c_4-\frac{3}{4}{c_2}^2\right)\right.$$
$$-\frac{3}{64}{c_1}^2\left(c_4+\frac{1}{12}{c_2}^2\right)$$
$$-\frac{31}{288}\left({c_1}^3-\frac{36}{31}c_1c_2+\frac{48}{31}c_3\right)$$
$$+\frac{1}{16}\left(c_2-\frac{15}{4}{c_1}^2\right)$$
$$+\frac{1}{768}{c_1}^4\left(c_2-\frac{5}{24}{c_1}^2\right)$$
$$+\frac{37}{768}c_1\left({c_1}^3+\frac{84}{37}c_1c_2-\frac{144}{37}c_3\right)$$
$$+\frac{1}{16}c_2\left(c_4-\frac{3}{4}{c_2}^2\right)+\frac{3}{128}{c_1}^5$$
$$\left.-\frac{1}{16}{c_3}^2+\frac{1}{72}\right],\qquad(4.19)$$

by applying the triangle inequality in (4.19) as well as Lemmas (2.1), (2.2), (2. 3), and (2.4), we get the desired inequality.

$$|H_{2,2}(F_f/2)|\leq\frac{1}{384}\cdot2(2+78+26)+\frac{1}{8}\cdot2+\frac{3}{64}\cdot4$$
$$\cdot2\left(1+\frac{1}{6}\right)$$

$$+\frac{31}{288}\left(2+\frac{52}{31}+\frac{86}{31}\right)+\frac{1}{16}\cdot2\left(\frac{15}{2}-1\right)+\frac{1}{768}\cdot16\cdot2$$
$$+\frac{37}{768}\cdot2\left(2+\frac{316}{37}+\frac{46}{37}\right)$$

$$+\frac{1}{16}\cdot2\cdot2+\frac{3}{128}\cdot32+\frac{1}{16}\cdot8+\frac{1}{72}$$
$$=\frac{53}{96}+\frac{1}{4}+\frac{7}{16}+\frac{25}{36}+\frac{13}{16}+\frac{1}{24}$$
$$+\frac{109}{96}+\frac{1}{4}$$

$$+\frac{3}{4}+\frac{1}{2}+\frac{1}{72}=\frac{529}{144}+\frac{127}{72}=\frac{87}{16}.$$

5. Third Hankel determinant for the Taylor coefficients of $f\in\mathcal{CST}_0(\sin z)$

Theorem 5.1.

If f is of the form (1.1) belongs to $\mathcal{CST}_0(\sin z)$, then

$$|H_{3,1}(f)|\leq\frac{32821+72\sqrt{138}}{138}.$$

Proof. In view of (3. 4), (3. 5), (3.6), we have

$$\begin{aligned} H_{2,2}(f) = a_2a_4 - {a_3}^2 & \\ &= -\frac{1}{96}{c_1}^4 + \frac{11}{24}{c_1}^3 + \frac{1}{4}{c_1}^2 + \frac{1}{4}c_1\,c_3 \\ &-\frac{5}{2}c_1\,c_2 - \frac{1}{4}{c_2}^2 - c_1 \end{aligned}$$

$$\begin{aligned} +c_2 + c_3 - 1 = \frac{1}{96}c_1\left(24c_3 - {c_1}^3\right) & \\ &+\frac{11}{24}\left({c_1}^3 - \frac{60}{11}c_1\,c_2 + \frac{24}{11}c_3\right) \\ &+\left(c_2 + \frac{1}{4}{c_1}^2\right) \\ &-\frac{1}{4}{c_2}^2 - c_1 - 1. \end{aligned}$$

Using the triangle inequality and Lemmas (2.1), (2.3), (2.4) and (2.5), we obtain

$$\begin{aligned} \left|H_{2,2}(f)\right| \le \frac{2}{96}\cdot 2\cdot 24\sqrt{\frac{24}{23}} & \\ &+\frac{11}{24}\left(2|1| + 2\left|\frac{60}{11} - 2(1)\right|\right. \\ &\left.+ 2\left|1 - \frac{60}{11} + \frac{24}{11}\right|\right) \end{aligned}$$

$$\begin{aligned} +2\left(1 + \frac{1}{2}\right) + \frac{1}{4}\cdot 4 + 2 + 1 & \\ &= \frac{2\sqrt{138}}{23} + \frac{11}{24}\left(2 + \frac{76}{11} + \frac{50}{11}\right) + 7 \\ &= \frac{2\sqrt{138}}{23} + \frac{37}{6} + 7 \\ &= \frac{1817 + 12\sqrt{138}}{138}. \qquad (5.1) \end{aligned}$$

$$\begin{aligned} |a_4 - a_2a_3| = \Bigg|\frac{5}{48}\left({c_1}^3 - \frac{24c_1c_2}{5} + \frac{24c_3}{5}\right) & \\ &+\left(c_2 - \frac{{c_1}^2}{2}\right) + \frac{3}{2}c_1 + 4 \\ &-\left(2 + \frac{1}{2}c_1\right) \end{aligned}$$

$$\begin{aligned} \left(c_1 + \frac{c_2}{2} - \frac{{c_1}^2}{4} + 3\right)\Bigg| & \\ &= \left|\frac{11}{48}\left({c_1}^3 - \frac{36c_1c_2}{11} + \frac{24c_3}{11}\right) - \frac{{c_1}^2}{2}\right. \\ &\left.- 2c_1 - 2\right| \end{aligned}$$

$$\begin{aligned} \le \frac{11}{48}\left(2|1| + 2\left|\frac{36}{11} - 2(1)\right| + 2\left|1 - \frac{36}{11} + \frac{24}{11}\right|\right) + 2 & \\ + 4 + 2 = \frac{13}{12} + 8 = \frac{109}{12}. \qquad (5.2) & \end{aligned}$$

Using the triangle inequality, Theorem (3.1), Corollary (3.1) and Eq (5.1) and (5.2), we obtain

$$\begin{aligned} \left|H_{3,1}(f)\right| \le |a_3|\left|H_{2,2}(f)\right| + |a_4||a_4 - a_2a_3| & \\ + |a_5|\left|H_{2,1}(f)\right| & \end{aligned}$$

$$\begin{aligned} \le 6\left(\frac{1817 + 12\sqrt{138}}{138}\right) + 10\cdot\frac{109}{12} + 17\cdot 4 & \\ &= \frac{1817 + 12\sqrt{138}}{23} + \frac{545}{6} + 68 \\ &= \frac{32821 + 72\sqrt{138}}{138}. \end{aligned}$$

6. Fourth Hankel determinant for the Taylor coefficients of $f \in \mathcal{CST}_0(\sin z)$

Theorem 6.1.

If f is of the form (1.1) belongs to $\mathcal{CST}_0(\sin z)$, then

$$\left|H_{4,1}(f)\right| \le \frac{7168484}{96} + \frac{1216\sqrt{138}}{23}.$$

Proof. From (1.9) and using the triangle inequality, we obtain

$$\begin{aligned} \left|H_{4,1}(f)\right| \le |a_7|\left|H_{3,1}(f)\right| + |a_6||\rho_1| + |a_5||\rho_2| & \\ + |a_4||\rho_3|. & \end{aligned}$$

We must find ρ_1, ρ_2 and ρ_3.

From (3. 4), (3.5), (3.6), and (3.7), we will find ρ_1

$$\begin{aligned} a_2a_5 - a_3a_4 = \frac{1}{4}c_1c_4 + \frac{9}{8}{c_1}^2c_2 - \frac{1}{8}{c_1}^3c_2 + \frac{1}{8}{c_2}^2c_1 & \\ -\frac{1}{8}{c_1}^2c_3 - \frac{3}{2}c_1c_2 + \frac{1}{96}{c_1}^5 & \end{aligned}$$

$$\begin{aligned} -\frac{1}{4}c_2c_3 - c_1c_3 + \frac{1}{2}c_3 - 2c_2 + \frac{13}{16}{c_1}^3 - \frac{13}{16}{c_1}^3 - 2c_1 & \\ -\frac{1}{2}{c_1}^2 - {c_2}^2 - 7 & \end{aligned}$$

$$\begin{aligned} = \frac{1}{4}c_1\left(c_4 - \frac{1}{2}c_1\,c_3\right) - \frac{13}{16}c_1\left({c_1}^3 - \frac{18}{13}c_1\,c_2 + \frac{16}{13}c_3\right) & \\ +\frac{13}{16}\left({c_1}^3 - \frac{24}{13}c_1\,c_2 + \frac{8}{13}c_3\right) & \\ -\frac{1}{8}c_2\left({c_1}^3 - c_1\,c_2 + 2c_3\right) & \\ -2\left(c_2 + \frac{1}{4}{c_1}^2\right) + \frac{1}{96}{c_1}^5 - 2c_1 & \\ -{c_2}^2 - 7. & \end{aligned}$$

Using the triangle inequality and Lemmas (2.1), (2.2), (2.3) and (2.4), we obtain

$$\begin{aligned} |a_2a_5 - a_3a_4| \le \frac{1}{4}\cdot 2\cdot 2 + \frac{13}{16} & \\ \cdot 2\left(2|1| + 2\left|\frac{18}{13} - 2(1)\right|\right. & \\ \left.+ 2\left|1 - \frac{18}{13} + \frac{16}{13}\right|\right) & \end{aligned}$$

$$\begin{aligned} +\frac{13}{16}\left(2|1| + 2\left|\frac{24}{13} - 2(1)\right| + 2\left|1 - \frac{24}{13} + \frac{8}{13}\right|\right) + \frac{1}{8} & \\ \cdot 2\left(2|1| + 2|1 - 2(1)|\right. & \end{aligned}$$

$$+2|1-1+2|)+2\cdot 2\left(1+\frac{1}{2}\right)+\frac{1}{96}\cdot 32+4+7$$
$$=1+\frac{13}{8}\left(2+\frac{16}{13}+\frac{22}{13}\right)$$
$$+\frac{13}{16}\left(2+\frac{4}{13}+\frac{6}{13}\right)+\frac{1}{4}(2+2+4)+6+\frac{1}{3}+11$$
$$=28+\frac{9}{4}+\frac{1}{3}=\frac{367}{12}. \qquad (6.1)$$

$$a_5-a_2a_4=\frac{1}{2}c_4+\frac{9}{16}{c_1}^2c_2-\frac{1}{4}{c_2}^2-\frac{3}{4}c_1\,c_3$$
$$-\frac{1}{12}{c_1}^4+\frac{1}{4}{c_1}^3-\frac{1}{2}c_1c_2-\frac{1}{2}c_2$$
$$-\frac{1}{2}{c_1}^2$$

$$-3c_1-3=-\frac{1}{12}c_1\left({c_1}^3-\frac{27}{4}c_1\,c_2+9c_3\right)$$
$$+\frac{1}{2}\left(c_4-\frac{1}{2}{c_2}^2\right)-\frac{1}{2}c_1\left(c_2-\frac{1}{2}{c_1}^2\right)$$
$$-\frac{1}{2}\left(c_2+{c_1}^2\right)+-3c_1-3.$$

Using the triangle inequality and Lemmas (2.1), (2.2), (2.3) and (2.4), we obtain

$$|a_5-a_2a_4|\le\frac{1}{12}$$
$$\cdot 2\left(2|1|+2\left|\frac{27}{4}-2(1)\right|\right.$$
$$\left.+2\left|1-\frac{27}{4}+9\right|\right)+\frac{1}{2}\cdot 2+\frac{1}{2}\cdot 2\cdot 2$$
$$+\frac{1}{2}\cdot 2(1+2)+3\cdot 2+3=\frac{1}{6}\left(2+\frac{19}{2}+\frac{13}{2}\right)+15$$
$$=3+15=18. \qquad (6.2)$$

From (3. 5), (3.6) and (3.7), we will find ρ_2.

$$H_{2,3}(f)=a_3a_5-a_4^2$$
$$=11c_1+\frac{1}{2}c_1c_4+\frac{1}{32}c_2c_4$$
$$+\frac{7}{32}{c_1}^2{c_2}^2-\frac{1}{8}{c_2}^3-\frac{1}{4}c_1\,c_2c_3$$
$$-\frac{1}{8}{c_1}^2c_4-\frac{3}{32}{c_1}^4c_2+\frac{1}{8}{c_1}^3c_3-\frac{1}{64}{c_3}^2+\frac{11}{24}{c_1}^3c_2$$
$$-\frac{3}{4}{c_2}^2c_1-\frac{3}{4}{c_1}^2c_3+\frac{1}{48}{c_1}^5$$
$$-\frac{11}{96}{c_1}^4-\frac{13}{16}{c_1}^2c_2-\frac{1}{2}c_1c_3-\frac{1}{2}c_1c_2-\frac{15}{8}{c_1}^3$$
$$-\frac{3}{2}{c_1}^2+7c_2+\frac{1}{4}c_2c_4+\frac{1}{2}c_2c_3$$
$$+\frac{1}{128}{c_1}^6+\frac{3}{2}c_4+3c_3+15$$
$$-\left(\frac{25}{2304}{c_1}^6+\frac{1}{4}{c_1}^2{c_2}^2-\frac{5}{48}{c_1}^4c_2\right.$$
$$+\frac{1}{4}{c_3}^2+\frac{5}{48}{c_1}^3c_3$$
$$-\frac{1}{2}c_1\,c_2c_3+{c_2}^2-\frac{5}{2}{c_1}^2c_2-\frac{3}{2}{c_1}^2+12c_1-c_1c_2$$
$$+8c_2-\frac{2}{3}{c_1}^3-\frac{1}{2}{c_2}^2c_1+2c_3$$
$$\left.+\frac{1}{4}{c_1}^3c_2+\frac{1}{2}c_2c_3+\frac{3}{4}c_1c_3-\frac{1}{4}{c_1}^2c_3+16\right)$$
$$=-c_1+\frac{1}{2}c_1c_4-\frac{1}{4}{c_1}^3c_2+\frac{1}{4}{c_2}^2c_1$$
$$-\frac{1}{4}{c_1}^2c_3+\frac{1}{8}{c_1}^5-\frac{41}{96}{c_1}^4+\frac{27}{16}{c_1}^2c_2-2c_1c_3$$
$$+\frac{1}{2}c_1c_2-\frac{29}{24}{c_1}^3+\frac{1}{4}{c_1}^2+7c_2$$
$$+\frac{1}{4}c_2c_4-\frac{1}{32}{c_1}^2{c_2}^2-\frac{1}{8}{c_2}^3+\frac{1}{4}c_1\,c_2c_3+\frac{1}{96}{c_1}^4c_2$$
$$-\frac{1}{2}c_2c_3-\frac{1}{8}{c_1}^2c_4+\frac{1}{48}{c_1}^3c_3$$
$$+\frac{3}{2}c_4-\frac{7}{2304}{c_1}^6-c_3-1-\frac{1}{4}{c_3}^2-{c_2}^2-8c_2$$
$$-\frac{1}{8}{c_2}^3$$

$$=-\frac{7}{2304}{c_1}^3\left({c_1}^3-\frac{24}{7}c_1c_2-\frac{48}{7}c_3\right)$$
$$+\frac{41}{96}c_1\left({c_1}^3+\frac{162}{41}c_1c_2-\frac{192}{41}c_3\right)$$
$$+\frac{1}{4}c_4\left(c_2+4{c_1}^2\right)$$
$$-\frac{1}{2}c_3\left(c_2+\frac{1}{2}{c_1}^2\right)$$
$$+7\left(c_2+\frac{7}{4}\,{c_1}^2\right)$$
$$-\frac{1}{8}{c_1}^2\left(c_4+\frac{1}{4}{c_2}^2\right)$$
$$-\frac{29}{24}\left({c_1}^3-\frac{12}{29}c_1c_2+\frac{24}{29}c_3\right)$$
$$-\frac{1}{4}{c_1}^3\left(c_2-\frac{1}{2}{c_1}^2\right)$$
$$-\frac{1}{4}c_3\left(c_3-\frac{1}{2}c_1\,c_2\right)$$
$$+\frac{3}{2}\left(c_4-\frac{2}{3}{c_2}^2\right)+\frac{1}{2}c_1\left(c_4+\frac{1}{2}{c_2}^2\right)$$
$$-c_1-8c_2-\frac{1}{8}{c_2}^3-1.$$

Using the triangle inequality and Lemmas (2.1), (2.2), (2.3) and (2.4), we obtain

$$\left|H_{2,3}(f)\right|=|a_3a_5-a_4^2|$$
$$\le\frac{7}{2304}\cdot 8\left(2+\frac{20}{7}+\frac{130}{7}\right)+\frac{41}{96}$$
$$\cdot 2\left(2+\frac{488}{41}+\frac{626}{41}\right)$$
$$+\frac{1}{4}\cdot 2\cdot 2(1+8)+\frac{1}{2}\cdot 2\cdot 2(1+1)+7\cdot 2\left(1+\frac{7}{2}\right)$$
$$+\frac{1}{8}\cdot 4\cdot 2\left(1+\frac{1}{2}\right)+\frac{1}{4}\cdot 8\cdot 2$$

$$+\frac{29}{24}\left(2+\frac{92}{29}+\frac{82}{29}\right)+\frac{1}{4}\cdot 2\ \cdot 2+\frac{3}{2}\cdot 2+\frac{1}{2}\cdot 2 \cdot 2(1+1)+2+8\cdot 2+\frac{1}{8}\cdot 8+1$$

$$=\frac{41}{72}+\frac{299}{12}+9+4+63+\frac{3}{2}+\frac{29}{3}+4+1+3+4+20=\frac{10415}{72}. \qquad (6.3)$$

Using the triangle inequality and Theorem (3.1), Corollary (3.1), (6.1) and (6.2), then

$$|\rho_1| = |a_3(a_2a_5-a_3a_4)-a_4(a_5-a_2a_4)+a_6(a_3-{a_2}^2\ )|$$

$$\leq |a_3||a_2a_5-a_3a_4|+|a_4||a_5-a_2a_4|+|a_6|\left|H_{2,1}(f)\right| \leq 6\cdot\frac{367}{12}\ +10\cdot 18+\frac{59}{2}\cdot 4=\frac{367}{2}+180+118=\frac{963}{2}. \qquad (6.4)$$

Using the triangle inequality and Theorem (3.1) and Eq (5.2), (6.2) and (6.3), then

$$|\rho_2| = |a_3(a_3a_5-a_4^2)-a_5(a_5-a_2a_4)+a_6(a_4-a_2a_3)| \leq |a_3||a_3a_5-a_4^2|$$

$$+|a_5||a_5-a_2a_4|+|a_6||a_4-a_2a_3| \leq 6\cdot\frac{10415}{72}+17\cdot 18+\frac{59}{2}\cdot\frac{109}{12}$$

$$=\frac{10415}{12}+306+\frac{6431}{24}=\frac{11535}{8}. \qquad (6.5)$$

Using the triangle inequality and Theorem (3.1) and Eq (5.1), (6.1) and (6.3), then

$$|\rho_3| = \left|a_4(a_3a_5-a_4^2)-a_5(a_2a_5-a_3a_4)+a_6H_{2,2}(f)\right| \leq |a_4||a_3a_5-a_4^2|$$

$$+|a_5||a_2a_5-a_3a_4|+|a_6|\left|H_{2,2}(f)\right| \leq 10\cdot\frac{10415}{72}+17\cdot\frac{367}{12}+\frac{59}{2}\left(\frac{1817+12\sqrt{138}}{138}\right)$$

$$=\frac{52075}{36}+\frac{6239}{12}+\frac{107203+708\sqrt{138}}{276}$$

$$=\frac{17698}{9}+\frac{107203+708\sqrt{138}}{276}=\frac{5849475+6372\sqrt{138}}{2484}. \qquad (6.6)$$

Using Theorem (3.1) and Eq (6.4), (6.5) and (6.6), we obtain

$$\left|H_{4,1}(f)\right| \leq |a_7|\left|H_{3,1}(f)\right|+|a_6||\rho_1|+|a_5||\rho_2|+|a_4||\rho_3| \leq \frac{313}{6}\left(\frac{32821+72\sqrt{138}}{138}\right)$$

$$+\frac{59}{2}\cdot\frac{963}{2}+17\cdot\frac{11535}{8}+10\left(\frac{5849475+6372\sqrt{138}}{2484}\right)=\frac{309729}{8}$$

$$+\frac{58494750+63720\sqrt{138}+30818919+67608\sqrt{138}}{2484}$$

$$=\frac{309729}{8}+\frac{89313669+131328\sqrt{138}}{2484}=\frac{7168484}{96}+\frac{1216\sqrt{138}}{23}.$$

7. Conclusion

This paper was inspired by a number of previous studies, we have obtained the upper bounds of some coefficient problems for functions in the class $\mathcal{CST}_0(\sin z)$including Taylor coefficients, logarithmic coefficients, and Hankel determinants of logarithmic coefficients. By employing subordination techniques and classical from geometric function theory. The results provided in this chapter perhaps could be the subject of further research related to the higher-order Hankel determinants of logarithmic coefficients and other coefficient problems, for instance, the Fekete-Szegö functional.

The obtained bounds enrich the theory of univalent functions and highlight the effectiveness of the adopted approach based on subordination and coefficient comparison.

Finally, the methods and results of this chapter open several directions for future research. These include the investigation of higher-order Hankel and Toeplitz determinants of logarithmic coefficients, study of Fekete-Szegö type problems for related to subclasses and the extension of the present analysis to other transcendental functions or different choices of subordinating functions. Such investigations may further deepen the understanding of the interplay between special functions and geometric properties of analytic functions.

References

[1] W. Janowski, "Some extremal problems for certain families of analytic functions," *Annales Polonici Mathematici*, vol. 28, no. 3, pp. 297–326, 1973.

[2] M. Reade, "On close-to-convex univalent functions," *Michigan Mathematical Journal*, vol. 3, pp. 59–62, 1955.

[3] P. L. Duren, *Univalent Functions*. New York, NY, USA: Springer-Verlag, 1983.

[4] M. Arif, M. Raza, H. Tang, S. Hussain, and H. Khan, "Hankel determinant of order three for familiar subsets of analytic functions related with sine function," *Open Mathematics*, vol. 17, no. 1, pp. 1615–1630, 2019.

[5] N. E. Cho, S. S. Kumar, and V. Ravichandran, "Radius problems for starlike functions associated with the sine function," *Bulletin of the Iranian Mathematical Society*, vol. 45, pp. 213–232, 2019, doi: 10.1007/s41980-018-0127-5.

[6] M. G. Khan, B. Ahmad, J. Sokol, Z. Muhammad, W. Mashwani, K. R. Chinram, *et al.*, "Coefficient problems in a class of functions with bounded turning associated with sine function," *European Journal of Pure and Applied Mathematics*, vol. 14, pp. 53–64, 2021, doi: 10.29020/nybg.ejpam.v14i1.3902.

[7] J. W. Noonan and D. K. Thomas, "On the second Hankel determinant of areally mean *p*-valent functions," *Transactions of the American Mathematical Society*, vol. 223, pp. 337–346, 1976, doi: 10.1090/S0002-9947-1976-0422607-9.

[8] B. Kowalczyk and A. Lecko, "Second Hankel determinant of logarithmic coefficients of convex and starlike functions of order alpha," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 45, no. 2, pp. 727–740, 2022.

[9] A. O. Lasode, A. O. Ajiboye, and R. O. Ayinla, "Some coefficient problems of a class of close-to-star functions of type α defined by means of a generalized differential operator," *International Journal of Nonlinear Analysis and Applications*, vol. 14, 2023, doi: 10.22075/IJNAA.2022.26979.3466.

[10] I. P. Kayumov, "On Brennan's conjecture for a special class of functions," *Mathematical Notes*, vol. 78, pp. 498–502, 2005.

[11] V. Allu, V. Arora, and A. Shaji, "On the second Hankel determinant of logarithmic coefficients for certain univalent functions," *Mediterranean Journal of Mathematics*, vol. 20, no. 2, Art. no. 81, 2023.

[12] R. M. El-Ashwah and D. K. Thomas, "Some subclasses of close-to-convex functions," *Journal of the Ramanujan Mathematical Society*, vol. 2, no. 1, pp. 85–100, 1987.

[13] T. H. MacGregor, "Majorization by univalent functions," *Duke Mathematical Journal*, vol. 34, 1967, doi: 10.1215/S0012-7094-67-03411-4.

[14] L. C. Ping and A. Janteng, "Subclass of starlike functions with respect to symmetric conjugate points," *International Journal of Algebra*, vol. 5, no. 16, pp. 755–762, 2011.

[15] V. Allu and V. Arora, "Second Hankel determinant of logarithmic coefficients for certain analytic functions," *Rocky Mountain Journal of Mathematics*, vol. 54, no. 2, pp. 343–359, 2024.

[16] S. K. Lee, K. Kanika, and V. Ravichandran, "Radius of starlikeness for classes of analytic functions," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 43, pp. 4469–4493, 2020, doi: 10.1007/s40840-020-01028-0.

[17] D. Mohamad, N. H. Abdul Wahid, and N. N. Hasni, "Coefficient problems for star-like functions with respect to symmetric conjugate points connected to the sine function," *European Journal of Pure and Applied Mathematics*, vol. 16, no. 2, pp. 1167–1179, 2023.

[18] S. O. Olatunji and S. Altinkaya, "Generalized distribution associated with quasi-subordination in terms of error function and Bell numbers," *Jordan Journal of Mathematics and Statistics*, vol. 14, 2021, doi: 10.47013/14.1.5.

[19] L. Shi, M. Arif, A. Rafiq, M. Abbas, and J. Iqbal, "Sharp bounds of Hankel determinant on logarithmic coefficients for functions of bounded turning associated with petal-shaped domain," *Mathematics*, vol. 10, no. 11, Art. no. 1939, 2022.

[20] D. K. Thomas, N. Tuneski, and N. A. Vasudevarao, *Univalent Functions: A Primer*. Berlin, Germany: Walter de Gruyter, 2018, doi: 10.1515/9783110560961-001.

[21] W. Ullah, S. N. Malik, D. Breaz, and L. I. Cotîrlă, "On logarithmic coefficients for starlike functions related to nephroid domain," *AIMS Mathematics*, vol. 10, no. 11, pp. 26905–26925, 2025, doi: 10.3934/math.20251183.

[22] I. A. Efraimidis, "Generalization of Livingston's coefficient inequalities for functions with positive real part," *Journal of Mathematical Analysis and Applications*, vol. 435, no. 1, pp. 369–379, 2016.

[23] W. C. Ma and D. Minda, "A unified treatment of some special classes of univalent functions," in *Conference Proceedings and Lecture Notes in Analysis*, Cambridge, MA, USA: International Press, 1992, pp. 157–169.

مقالة بحثية

المعاملات اللوغاريتمية ومحددات هانكل لصنف جزئي جديد من الدوال القريبة من النجمية المتعلقة بدالة الجيب

صفاء حمود محمد السقاف[1]،*

[1] *قسم الرياضيات، كلية صبر للعلوم الإنسانية والتطبيقية، جامعة لحج، اليمن*

*** الباحث الممثّل: صفاء حمود محمد السقاف؛ البريد الالكتروني: safaalsaqqaf1@gmail.com**

استلم في: 01 يونيو 2026 / قبل في: 16 يونيو 2026 / نشر في 30 يونيو 2026

المُلخّص

تبحث هذه الورقة المعاملات اللوغاريتمية لصنف جزئي جديد من الدوال القريبة من النجمية، نحن نشتق صيغ صريحة للمعاملات اللوغاريتمية من γ_1 إلى γ_6 للدوال في هذا الصنف. أيضًا نحن نثبت الحدود الدقيقة لمعاملات هانكل، محددات هانكل، $H_{2,1}(f)$، $H_{2,2}(f)$، $H_{3,1}(f)$ و $H_{4,1}(f)$ المرتبطة بالصنف $\mathcal{CST}_0(\sin z)$. بالإضافة الى ذلك، نحن نستنتج التقديرات الدقيقة لمحددات هانكل للمعاملات اللوغاريتمية $H_{2,1}(F_f/2)$ و $H_{2,2}(F_f/2)$ ضمن نفس الصنف.

الكلمات المفتاحية: الدوال التحليلية؛ الدوال القريبة من النجمية؛ معاملات تايلور؛ المعاملات اللوغاريتمية؛ التبعية؛ محددة هانكل.

How to cite this article:

S. H. M. Al-Saqqaf, “LOGARITHMIC COEFFICIENTS AND HANKEL DETERMINANT FOR A NEW SUBCLASS OF CLOSE-TO-STAR FUNCTIONS ASSOCIATED WITH THE SINE FUNCTION”, *Electron. J. Univ. Aden Basic Appl. Sci.*, vol. 7, no. 2, pp. 219-236, Jun. 2026. DOI: https://doi.org/10.47372/ejua-ba.2026.2.545